Newton-Mechanik

$\vec{r}(t)$, m_0, $\vec{F}(\vec{r},t)$

$$\boxed{\frac{d^2}{dt^2}\vec{r}(t) = \frac{1}{m_0}\vec{F}(\vec{r},t), \quad \vec{r}(0)=\vec{r}_0, \quad \left.\frac{d\vec{r}}{dt}\right|_{t=0} = \vec{v}_0 \quad \text{Eindeutige Trajektorie}}$$

Tangentialvektor $\hat{t} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{d\vec{r}(s)}{ds}$ Geschwindigkeit $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \frac{ds}{dt}\frac{d\vec{r}(s)}{ds}$

Bahnlänge $s(t) = \int_0^t dt' |v(t')|$ $\frac{ds}{dt} = |v(t)|$ Beschleunigung $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}$

Krümmung $\kappa(s) = \left|\frac{d^2\vec{r}(s)}{ds^2}\right| = \left|\frac{d}{ds}\left(\frac{\vec{v}(t)}{|\vec{v}(t)|}\right)\right| = \left|\frac{dt}{ds}\right|\left|\frac{d}{dt}\left(\frac{\vec{v}(t)}{|\vec{v}(t)|}\right)\right| = \left|\frac{1}{|\vec{v}(t)|}\right|\left|\frac{\vec{a}}{|\vec{v}|} + \vec{v}\frac{d}{dt}\left(\frac{1}{|\vec{v}|}\right)\right|$

$= \frac{1}{|\vec{v}|^2}\sqrt{\vec{a}^2\vec{v}^2 - (\vec{v}\cdot\vec{a})^2}$

Normalenvektor $\hat{n}(s) = \frac{1}{\kappa(s)}\frac{d\hat{t}}{ds}$ $\kappa(s) = \left|\frac{d^2\vec{r}(s)}{ds^2}\right|$

Binormalenvektor $\hat{b}(s) = \hat{t}(s)\times\hat{n}(s)$ Torsion $\tau(s) = \left|\frac{d\vec{b}(s)}{ds}\right|$

Mathematik der Felder

$\phi: \mathbb{R}^4 \to \mathbb{R}$ $\phi(\vec{r},t)$ Skalarfelder

$$\left.\frac{d\phi}{dt}\right|_{\vec{r}(t)} = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial\phi}{\partial t} = \vec{\nabla}\phi\cdot\frac{d\vec{r}}{dt} + \partial_t\phi$$

$d\phi(x,y,z) = \partial_x\phi\,dx + \partial_y\phi\,dy + \partial_z\phi\,dz$ Totales Differential

$\vec{F}: \mathbb{R}^4 \to \mathbb{R}^3$ $\vec{F}(\vec{r},t)$ Vektorfelder

$W = \int_C \vec{F}\cdot d\vec{s} = \int_{t_1}^{t_2} \vec{F}(\vec{r}(t))\cdot\left(\frac{d\vec{r}}{dt}\right)dt$ Arbeit im Kraftfeld

$$\operatorname{rot}\vec{F} = \vec{\nabla}\times\vec{F} = \varepsilon_{ijk}e_i\partial_j F_k = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$

auf einfach zsh. Gebieten

$W = U_2 - U_1$ wegunabh. $\Leftrightarrow$ $\operatorname{rot}\vec{F} = 0$ $\Leftrightarrow$ $\vec{F}$ konservativ: $\exists U: \vec{F} = \vec{\nabla}U$

Parametrisierte Fläche $\subset\mathbb{R}^3$: $\vec{r} = \begin{pmatrix} u \\ v \\ f(u,v)\end{pmatrix}$ Normale $\hat{n} = \frac{\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|}$

Fluss $\Phi = \iint_A \vec{F}\cdot d\vec{A} = \int_{u_1}^{u_2}du\int_{v_1}^{v_2}dv\,\underbrace{\left\langle \vec{F}(\vec{r}(u,v)), \frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right\rangle}_{=} \begin{vmatrix} F_1\circ\vec{r} & F_2\circ\vec{r} & F_3\circ\vec{r} \\ \partial_u\vec{r}_1 & \partial_u\vec{r}_2 & \partial_u\vec{r}_3 \\ \partial_v\vec{r}_1 & \partial_v\vec{r}_2 & \partial_v\vec{r}_3 \end{vmatrix}$

Harmonischer Oszillator

$x(0) = x_0$ $\dot{x}(0) = v_0$ Anfangsbed.

$\frac{d^2}{dt^2}x(t) + 2\gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = a(t)$ DGL $\omega_0^2 = \frac{D}{m}$ Feder

$\Rightarrow x(t) = x_h(t) + x_p(t)$ $\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right)x_h(t) = 0$ $\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right)x_p(t) = a(t)$

Homogene Lsg. $x_h(t) = A\,x^{(1)}(t) + B\,x^{(2)}(t)$

(1) Ungedämpft: $\gamma = 0$: $\ddot{x} + \omega_0^2 x = 0 \Rightarrow x = e^{\mu t}$ $\mu_{1,2} = \pm i\omega_0 \Rightarrow$ $x^{(1)}(t) = \cos(\omega_0 t)$, $x^{(2)}(t) = \sin(\omega_0 t)$

$\Rightarrow x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t + \varphi)$

(2) Gedämpft: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \Rightarrow x = e^{\mu t}$ $\mu_{1,2} = -\gamma \pm i\overbrace{\sqrt{\omega_0^2-\gamma^2}}^{\Omega} = -\gamma \pm \Omega$

$\Rightarrow x^{(1)}(t) = e^{-\gamma t}e^{i\Omega t}$, $x^{(2)}(t) = e^{-\gamma t}e^{-i\Omega t}$

$\rightarrow$ Schwingfall: $\omega_0 > \gamma$: $\Omega = \sqrt{\omega_0^2-\gamma^2} \neq \omega_0 \Rightarrow x(t) = \left(Ae^{i\Omega t} + A^* e^{-i\Omega t}\right)e^{-\gamma t}$

$\rightarrow$ Kriechfall: $\omega_0 < \gamma$: $\Gamma = \sqrt{\gamma^2-\omega_0^2} \Rightarrow x(t) = e^{-\gamma t}\left(Ae^{\Gamma t} + Be^{-\Gamma t}\right)$

$\Rightarrow x(t) = e^{-\gamma t}\left(C_1\cosh(\Gamma t) + C_2\sinh(\Gamma t)\right)$

$\rightarrow$ Aperiodischer Grenzfall: $\omega_0^2 = \gamma^2$: $x(t) = Ae^{-\gamma t} + Bte^{-\gamma t}$

Partikuläre Lsg. $x_p(t) = \int_{-\infty}^{\infty} G(t-t')a(t')$ Green'sche Funktion

$\underbrace{\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right)}_{D} G(t-t') = \delta(t-t') \Rightarrow$ $DG(t-t') = 0 \;\forall t \neq t'$, $\int_I DG(t-t')dt' = 1 \;\forall I \ni t'$

$G(t-t') = g(t) = \begin{cases} 0, & t<0 \\ 0, & t=0 \\ \neq 0, & t>0 \end{cases}$ $g(t) = Ce^{\mu_1 t} - Ce^{\mu_2 t}$

$$g(t) = \begin{cases} \Theta(t)\frac{e^{-\gamma t}}{\sqrt{\omega_0^2-\gamma^2}}\sin(\Omega t), & \omega_0 > \gamma \quad \Omega = \sqrt{\omega_0^2-\gamma^2} \\ \Theta(t)\frac{e^{-\gamma t}}{\sqrt{\omega_0^2-\gamma^2}}\sinh(\Gamma t), & \omega_0 < \gamma \quad \Gamma = \sqrt{\gamma^2-\omega_0^2} \\ \Theta(t)\cdot te^{-\gamma t}, & \omega_0 = \gamma \end{cases}$$

$x(t) = x_h(t) + \int_{-\infty}^{\infty}dt'\,g(t-t')a(t')$

Dynamik (Newton)

Gleichgewichtslagen in konservativen Kraftfeldern $\vec{F} = \vec{\nabla}U$

(meta-) stabiles Glgw. $\Leftrightarrow$ (lokales) Minimum von U

instabil $\Leftrightarrow$ Maximum

$U(\vec{r}) = U(\vec{r}_0) + \vec{\nabla}_{\vec{r}_0}U(\vec{r}-\vec{r}_0) + \frac{1}{2}(\vec{r}-\vec{r}_0)D^2U(\vec{r}-\vec{r}_0)^T$ Taylor $F = D^2U(x-x_0)$

SRT

Zweiteilchenproblem und Zentralkraft

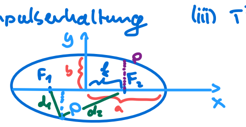
Kepler:

(i) Bewegung von Planeten in Ebene (Ellipse/Hyperbel)

(ii) Flächensatz/Drehimpulserhaltung (iii) $T^2/a^3 = \text{const.}$

Ellipsen:

$\varepsilon = \frac{f}{a}$ $\frac{x^2}{a} + \frac{y^2}{b} = 1$



$p^2 = b^2(1-\varepsilon^2)$ $p = a(1-\varepsilon^2)$ $\varepsilon = \frac{\sqrt{a^2-b^2}}{a}$

Hyperbelförmige Bahnen: $p^2 = b^2(\varepsilon^2-1)$ $p = a(\varepsilon^2-1)$ $\varepsilon = \frac{\sqrt{a^2+b^2}}{a}$

$\varepsilon = \frac{f}{a} > 1$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x = f + r\cos\varphi$, $y = r\sin\varphi$ $r(\varphi) = \frac{p}{1+\varepsilon\cos\varphi}$ $d_1 + d_2 = 2a$ Polarkoordinaten (Ursprung in F_2)

Zweiteilchenproblem $\vec{F}_{12} = -\vec{F}_{21}$ $\frac{d\vec{P}}{dt} = 0$ $\frac{d\vec{L}}{dt} = 0$ $\frac{dE}{dt} = 0$ $\vec{F}_{12} = -\vec{\nabla}_{\vec{r}_2}U(|\vec{r}_1-\vec{r}_2|)$

Schwerpunktkoordinaten $\vec{r} = \vec{r}_1 - \vec{r}_2$ $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} \Rightarrow$ $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $M = m_1 + m_2$

$\vec{P} = M\dot{\vec{R}} \Rightarrow \dot{\vec{P}} = 0$ $\frac{m_1 m_2}{M}\ddot{\vec{r}} = \mu\ddot{\vec{r}} = \vec{F}_{21}$ $\vec{L} = \underbrace{\vec{R}\times(M\dot{\vec{R}})}_{L_{SP}} + \underbrace{\vec{r}\times(\mu\dot{\vec{r}})}_{L_{rel}}$ $\dot{\vec{L}} = 0$

$E = \frac{1}{2}M(\dot{\vec{R}})^2 + \frac{1}{2}\mu(\dot{\vec{r}})^2 + U(\vec{r}) = E_{SP} + E_{rel}$

Keplerproblem $L_{rel} = L_z\hat{e}_z = \mu r^2\dot{\varphi} = \text{const.}$ $\vec{F}_{21} = -G\frac{m_1 m_2}{r^2}\hat{e}_r$ $U = -\frac{\alpha}{r}$ $\alpha = m_1 m_2 G$

$E_{rel} = \frac{1}{2}\mu(\dot{r})^2 + \frac{L_z^2}{2\mu r^2} + U(r)$ $\frac{dA}{dt} = \frac{L_z}{2\mu}$ $T^2 = 4\pi^2\frac{\mu}{\alpha}a^3$

Fourier-Transformation

$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$

$f(t) \in L_1(\mathbb{C}) \rightsquigarrow \tilde{f}(\omega) \in L_1(\mathbb{C})$ Funktional $\mathcal{F}\{f(t)\}(\omega)$

$\tilde{f}(\omega) = \mathcal{F}\{f(t)\}(\omega) = \int_{-\infty}^{\infty}dt\,f(t)e^{i\omega t}$ $f(t) = \mathcal{F}^{-1}\{\tilde{f}(\omega)\}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\tilde{f}(\omega)e^{-i\omega t}$

Eigenschaften:

$\mathcal{F}\{\alpha f(t) + \beta g(t)\}(\omega) = \alpha\mathcal{F}\{f(t)\}(\omega) + \beta\mathcal{F}\{g(t)\}(\omega)$

$\mathcal{F}\{f(t-t_0)\}(\omega) = e^{i\omega t_0}\mathcal{F}\{f(t)\}(\omega)$

$\mathcal{F}\{f * g(t)\}(\omega) = \mathcal{F}\{f(t)\}(\omega)\cdot\mathcal{F}\{g(t)\}(\omega) = \tilde{f}(\omega)\tilde{g}(\omega)$

$\mathcal{F}\{\frac{d^n}{dt^n}f(t)\}(\omega) = (-i\omega)^n\mathcal{F}\{f(t)\}(\omega) = (-i\omega)^n\tilde{f}(\omega)$

$f(t) \in \mathbb{R} \Leftrightarrow \tilde{f}(-\omega) = \tilde{f}^*(\omega)$

$f(-t) = \pm f(t) \Rightarrow \tilde{f}(-\omega) = \pm\tilde{f}(\omega)$

Beispiele:

$f(t) = \delta(t) \Rightarrow \tilde{f}(\omega) = 1$

$f(t) = e^{-t^2/\tau^2} \Rightarrow \tilde{f}(\omega) = \tau\sqrt{\pi}e^{-\frac{\omega^2\tau^2}{4}}$

$f(t) = \Theta(t+\frac{\tau}{2}) - \Theta(t-\frac{\tau}{2}) \Rightarrow \tilde{f}(\omega) = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$

Lösung des H.O.:

$\mathcal{F}\left\{\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right)x_p(t)\right\}(\omega) = \tilde{a}(\omega)$

$[-\omega^2 - 2i\gamma\omega + \omega_0^2]\,\tilde{x}_p(\omega) = \tilde{a}(\omega) \Rightarrow \tilde{x}_p(\omega) = \frac{\tilde{a}(\omega)}{\omega_0^2 - 2i\gamma\omega - \omega^2}$

$x_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,\frac{\tilde{a}(\omega)}{\omega_0^2 - 2i\gamma\omega - \omega^2}e^{-i\omega t} \Rightarrow \tilde{G}(\omega) = \frac{1}{\omega_0^2 - 2i\gamma\omega - \omega^2}$

$\tilde{G}(\omega) = \frac{1}{\sqrt{(\omega_0^2-\omega^2) + 4\gamma^2\omega^2}}\exp\left(i\arctan\left(\frac{2\gamma\omega}{\omega_0^2-\omega}\right)\right)$

$\omega_{max} = \sqrt{\omega_0^2 - 2\gamma^2}$

Lagrange-Formalismus

1. Art

N Massepunkte, N_z Zwangsbedingungen $\quad f=3N$

holonom: keine Abh. von $\dot{\vec r}_i$

rheonom: $\partial_t A_\mu \neq 0$ sonst skleronom

$$A_\mu(\vec r_1,\dots,\vec r_N,t)=0\,,\quad \mu=1,\dots,N_z$$

$\Rightarrow$ Zwangskraft $\quad \vec Z_i=\sum_{\mu=1}^{N_z}\lambda_\mu^{(i)}\vec\nabla_{\vec r_i}A_\mu(\dots)$

$$\frac{d}{dt}(m\dot{\vec r}_i)=\vec F_i+\sum_{\mu=1}^{N_z}\lambda_\mu(t)\,\vec\nabla_{\vec r_i}A_\mu(\vec r_1,\dots,\vec r_N,t)\qquad i=1,\dots,N\quad \text{Bewegungsgln.}$$

Zwangskräfte leisten keine Arbeit

N Massep., N_z Zwangsbed. $\Rightarrow f=3N-N_z$ $\quad A_\mu(\vec r_1,\dots,\vec r_N)=0$ $\qquad$ 2. Art

Verallgemeinerte Koordinaten: $q_1(t),\dots,q_f(t)$ $\quad \vec r_i=\vec r_i(q_1,\dots,q_f,t)$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q_\alpha}\right)=\frac{\partial L}{\partial q_\alpha}}\quad \text{für } \alpha=1,\dots,f\qquad \underbrace{L}_{\text{Lagrangian}}=L(\{q_\alpha\},\{\dot q_\alpha\},t)=T(\{q_\alpha\},\{\dot q_\alpha\},t)-U(\{q_\alpha\},t)$$

für holonome, skleronome Zwangsbedingungen

Erhaltungsgrößen:

$$\frac{\partial L}{\partial t}=0\Rightarrow\frac{d}{dt}\left\{\sum_{\alpha=1}^{f}\frac{\partial L}{\partial\dot q_\alpha}\dot q_\alpha-L\right\}=\frac{d}{dt}(T+U)=0\quad\text{Energieerhaltung}$$

Zyklische Koordinaten:

$$\frac{\partial L}{\partial q_\alpha}=0\ \ \text{verallgemeinerte Kraft}\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q_\alpha}\right)=0\ \ \text{verallgemeinerter Impuls}$$

$\Rightarrow$ H erhalten

Bsp.: Ebene, Winkel $\varphi\Rightarrow L_z=\frac{\partial L}{\partial\dot\varphi}=ml^2\dot\varphi$

Nicht-konservative Kräfte $\Rightarrow \dot{\vec r}$-Abh. im Potential.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_\alpha}-\frac{\partial U}{\partial\dot q_\alpha}\right)=\frac{\partial L}{\partial q_\alpha}$$

SBB $\leftrightarrow$ CFF FFS

Reibungskräfte:

Dissipationsfkt. $\quad F(\dot{\vec r})=\sum_{i=1}^{N}\frac12\gamma_i(\dot{\vec r}_i)^2$

Unbestimmtheit von L:

$$L^*(q,\dot q,t)=L(q,\dot q,t)+\frac{d}{dt}f(q,t)$$

$$\Delta S=f(q(t_2),t_2)-f(q(t_1),t_1)$$

$$\Rightarrow\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}+\frac{\partial F}{\partial\dot q_\alpha}=0$$

Ergänzung

Variationsrechnung

Suche Minimum von $\quad J[y]=\int_{x_1}^{x_2}dx\,F\left(y,\frac{dy}{dx},x\right)$

Randbed.: $y(x_1)=y_1$, $y(x_2)=y_2$

$$y_0\ \text{Minimum}\iff\frac{\partial F}{\partial y}(y_0,y_0',x)-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}(y_0,y_0',x)\right)=0$$

$$\iff\frac{\delta J[y]}{\delta y}=\left(\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}\right)[y]=0$$

Nebenbedingungen:

$$J[y]=\int_{x_1}^{x_2}F(y,y',x)\,dx\quad\text{unter NB}\quad K[y]=\int_{x_1}^{x_2}G(y,y',x)\,dx=C$$

$$\Rightarrow J^*[y]=J[y]-\lambda(K[y]-C)$$

$$\left\{\begin{aligned}\frac{\delta J^*[y]}{\delta y}&=0\\ \frac{\partial J^*}{\partial\lambda}&=C\end{aligned}\right\}=\left\{\begin{aligned}&\frac{\partial F^*}{\partial y}-\frac{d}{dx}\left(\frac{\partial F^*}{\partial y'}\right)=0\\ &K[y]=C\end{aligned}\right\}$$

mit $\quad F^*(y,y',x)=F(y,y',x)-\lambda G(y,y',x)$

$$\frac{\delta J^*}{\delta y}=\frac{\delta J}{\delta y}-\lambda\frac{\delta K}{\delta y}=0$$

λ durch Randbedingungen bestimmen

Prinzip der kleinsten Wirkung:

Wirkung $\quad S[\{q_\alpha\}]=\int_{t_1}^{t_2}dt\,L(\{q_\alpha\},\{\dot q_\alpha\},t)$

$$\frac{\delta S}{\delta q_\alpha}\overset{!}{=}0\iff\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q_\alpha}\right)-\frac{\partial L}{\partial q_\alpha}=0\,,\qquad\alpha=1,\dots,f$$

Lagrange erster Art:

$$S^*[\{\vec r_i\}]=\int_{t_1}^{t_2}dt\left(L(\{\vec r_i\},\{\dot{\vec r}_i\},t)+\sum_{\mu=1}^{N_z}\lambda_\mu(t)A_\mu(\{\vec r_i\},\{\dot{\vec r}_i\},t)\right)$$

$x_i\to x_i^*=x_i+\varepsilon\psi_i(\{x_i\},\{\dot x_i\},t)$ $\quad t\to t^*=t+\varepsilon\phi(\{x_i\},\{\dot x_i\},t)$ $\quad$ Noether

Wenn $S^*[\{x_i^*\}]=\int_{t_1^*}^{t_2^*}dt^*L^*(\{x_i^*\},\{\dot x_i^*\},t)\overset{!}{=}S[\{x_i\}]$, dann:

Noether-Ladung $\quad Q(\{x_i\},\{\dot x_i\},t)=\sum_i\frac{\partial L}{\partial\dot x_i}\cdot\psi_i+\left(L-\sum_i\frac{\partial L}{\partial\dot x_i}\dot x_i\right)\phi\iff\underbrace{\frac{dQ}{dt}=0}_{\text{Erhaltungsgr.}}$.

Erweitert: Wenn $\delta S^*=\delta S\iff S^*=S+\underbrace{\Delta S}_{\text{const.}}$: $\quad Q=\sum_i\frac{\partial L}{\partial\dot x_i}\psi_i+\left(L-\sum_i\frac{\partial L}{\partial\dot x_i}\dot x_i\right)\phi-f(\{x_i\},t)$

Bewegte Bezugssysteme

Inertialsystem (IS) $\quad (x,y,z)\equiv(x_1,x_2,x_3)$

Koordinatensystem (KS') $\quad (x',y',z')\equiv(x_1',x_2',x_3')=(x_1,x_2,x_3)-d(t)$

$\vec r=\vec r\,'+d(t)$, $\quad\frac{d}{dt}(m\dot{\vec r})\overset{!}{=}\frac{d}{dt}(m\dot{\vec r}\,')\Rightarrow\ddot{\vec d}(t)=0\Rightarrow\vec d(t)=\vec vt+\vec r_0$ $\quad$ Galilei

Bei Drehungen:

$\vec r\,'=D\vec r\qquad x_i'=\sum_{j=1}^{3}D_{ij}x_j\qquad D^TD=1\qquad\Rightarrow D_{ij}=\text{const.}$

Allg. Galilei-Trafo:

$$x_i'=\sum_j D_{ij}x_j+v_it+r_{0,i}\,,\qquad t'=t-t_0\,,\qquad D^TD=1$$

Beschleunigte Bezugssysteme

Lineare Beschleunigung: $\vec d(t)=\frac12\vec bt^2$

$\vec r(t)=\vec r\,'(t)+\frac12\vec bt^2\qquad\dot{\vec r}(t)=\dot{\vec r}\,'(t)+\frac12\vec bt$

$\frac{d}{dt}(m\dot{\vec r})=0\Rightarrow\frac{d}{dt}(m\dot{\vec r}\,')+m\vec b=0$

Drehung:

$D=D(\hat z,\phi)\,D(\hat x,\theta)\,D(\hat z,\psi)$

$$=\begin{pmatrix}\cos\phi&-\sin\phi&0\\ \sin\phi&\cos\phi&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}\begin{pmatrix}\cos\psi&-\sin\psi&0\\ \sin\psi&\cos\psi&0\\0&0&1\end{pmatrix}$$

$\dot{\vec r}=\dot{\vec r}\,'+\vec\omega\times\vec r\,'$

$$\Rightarrow m\ddot{\vec r}\,'=\underbrace{-2m\vec\omega\times\dot{\vec r}\,'}_{\text{Coriolis}}\underbrace{-m\vec\omega\times(\vec\omega\times\vec r\,')}_{\text{Zentrifugal}}\underbrace{-m(\dot{\vec\omega}\times\vec r\,')}_{\text{Trägheit}}$$

Starrer Körper

N Massepunkte, verbunden

$$\Theta_{ij}=\sum_{n=1}^{N}\left(\vec r_n^{\,2}\delta_{ij}-x_i^{(n)}x_j^{(n)}\right)\qquad\Theta=\sum_{i=1}^{N}\left(\vec r_i^{\,2}\,\mathrm{id}-\vec r_i\otimes\vec r_i\right)$$

$$T=T_{trans}+T_{rot}=\frac12MV^2+\frac12\Theta\omega^2\qquad T_{rot}=\frac12\Theta\omega^2=\frac12\vec\omega\cdot\Theta\cdot\vec\omega^T=\frac12(\Theta\cdot\vec\omega)\cdot\vec\omega$$

$\vec L=\Theta\cdot\vec\omega\qquad$ Eigenwerte von Θ: Hauptachsen

Im Hauptachsensystem:

$\Theta_1\dot\omega_1+(\Theta_3-\Theta_2)\omega_2\omega_3=M_1\qquad m\ddot x_1=F_1$

$\Theta_2\dot\omega_2+(\Theta_1-\Theta_3)\omega_3\omega_1=M_2\qquad m\ddot x_2=F_2$

$\Theta_3\dot\omega_3+(\Theta_2-\Theta_1)\omega_1\omega_2=M_3\qquad m\ddot x_3=F_3$

Hamilton-Mechanik

$p_\alpha=\frac{\partial L}{\partial\dot q_\alpha}\longrightarrow\dot q_\alpha=\dot q_\alpha(\{q_\alpha\},\{p_\alpha\})$

Hamiltonian $\quad H(\{q_\alpha\},\{p_\alpha\},t)=\sum_\alpha\dot q_\alpha(\{q_\alpha\},\{p_\alpha\},t)\cdot p_\alpha-L(\{q_\alpha\},\{\dot q_\alpha(\{q_\alpha\},\{p_\alpha\},t)\},t)$

Legendre-Transformation: $\quad H=\sum_\alpha\frac{\partial L}{\partial\dot q_\alpha}\dot q_\alpha-L\qquad$ nicht mehr von $\dot q_\alpha$ abh.

Kanonische Gleichungen:

$$\boxed{\dot q_\alpha=\frac{\partial H}{\partial p_\alpha}\,,\quad\dot p_\alpha=-\frac{\partial H}{\partial q_\alpha}=\{H,p_\alpha\}\,,\quad\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}\,,\quad\alpha=1,\dots,f}$$

Poisson-Klammern:

$$f=f(\{q_\alpha\},\{p_\alpha\},t)$$

$$\Rightarrow\frac{df}{dt}=\frac{\partial f}{\partial t}+\{H,f\}\qquad\Rightarrow\ \dot q_\alpha=\{H,q_\alpha\},\ \ \dot p_\alpha=\{H,p_\alpha\}$$

$$\{g,f\}:=\sum_\alpha\left(\frac{\partial f}{\partial q_\alpha}\frac{\partial g}{\partial p_\alpha}-\frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial q_\alpha}\right)$$

$\{q_\alpha,q_\beta\}=0$

$\{p_\alpha,p_\beta\}=0$

$\{f,g\}=-\{g,f\}$

$\{p_\alpha,q_\beta\}=\delta_{\alpha\beta}$

$\{f,c\}=0\quad$ für $c=\text{const.}$

$\{q_\alpha q_\beta,p_\gamma\}=q_\alpha\{q_\beta,p_\gamma\}+\{q_\alpha,p_\gamma\}q_\beta$

$\{f_1+f_2,g\}=\{f_1,g\}+\{f_2,g\}$

Poissonsches Theorem:

f,g Erhaltungsgrößen

$\Rightarrow\{H,f\}=0\,,\ \{H,g\}=0\qquad\Rightarrow\{H,\{f,g\}\}=0$

Kanonische Transformationen:

Koord., Impulse nicht eindeutig:

$Q_i=Q_i(\{q_\alpha\},\{p_\alpha\},t)\qquad\dot Q_i\overset{!}{=}\frac{\partial H'}{\partial P_i}$

$P_i=P_i(\{q_\alpha\},\{p_\alpha\},t)\qquad\dot P_i=-\frac{\partial H'}{\partial Q_i}$

$$\left.\begin{aligned}&H=H(\{q_i\},\{p_i\},t)\longrightarrow L=\sum p_i\dot q_i-H\\ &H'=H'(\{Q_i\},\{P_i\},t)\longrightarrow L'=\sum P_i\dot Q_i-H'\end{aligned}\right\}\Rightarrow L-L'=\frac{d}{dt}F(\{q_i\},\{Q_i\},t)$$

Fkt. F erzeugt kanon. Trafo.

Liouville: Integral über Volumen im Phasenraum invariant unter kanon. Trafos.