Finance: Lecture 5 - Factor models
Chapters 6,7,13 of DD, Chapters 8-9 of Cochrane (2001)

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March 2010

This class

1. Generic factor models
2. Example: the CAPM
3. Conditional vs. unconditional models
4. The Arbitrage Pricing Theory

A generic factor model

We pose that expected returns are determined by:

$$\mathrm{E}[R_j] = \gamma + \beta_j \lambda$$

where γ and the vector λ are/contain coefficients and β is determined by the regression:

$$R_{j,t} = a_j + \beta_j F_t + \epsilon j, t$$

where F is a vector of random risk factors.
Why is there "substance" to this model?

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Why is there "substance" to this model?

The regression model allows for cross-sectional variation in the intercept. The coefficient γ will be the same for all assets.

If there is a risk-free asset, the return of this asset will have zero exposure to any risk-factors and $\gamma = r_f$.

Factor representation of pricing kernels

We have already seen a factor representation of the CCAPM:

$$ER_j = \gamma + \beta_{j,M}\lambda_M$$

How can we, more generally, relate pricing kernels and factor models?
Proposition: Given the model

$$M = a + b(F - \mathrm{E}[F])$$

we can find γ and λ such that

$$\mathrm{E}[R_j] = \gamma + \beta_j\lambda$$

where β_j are coefficients of a regression of R_j on a constant and the factors F. Conversely, given γ and λ, we can find a and b such that $M = a + b(F - \mathrm{E}[F])$.

Proof

Since

$$\mathrm{E}[R_j] = \frac{1}{\mathrm{E}[M]} - \frac{\mathrm{Cov}[M, R_j]}{\mathrm{E}[M]},$$

$M = a + b(F - \mathrm{E}[F])$ implies that

$$\begin{aligned} \mathrm{E}[R_j] &= \frac{1}{a} - \frac{\mathrm{E}[R_j F]b}{a} \\ &= \frac{1}{a} - \frac{\mathrm{E}[R_j F](\mathrm{E}[FF])^{-1}\mathrm{E}[FF]b}{a} \\ &= \frac{1}{a} - \frac{\beta_j \mathrm{E}[FF]b}{a} = \gamma + \beta_j \lambda \end{aligned}$$

(because $\beta_j = (\mathrm{E}[FF])^{-1})\mathrm{E}[FR_j]$).

This class

The CAPM

A "capital asset pricing model" (CAPM) is a model in which:

$$M_{t+1} = a_t + b_t R_{W,t+1}$$

Such a model can be derived in a number of models. We will cover:

- A two period quadratic utility model
- Rubinstein's (1976) log utility model
- Merton's (1973) intertemporal capital asset pricing model ICAPM

A two period quadratic utility model

Investors maximize:

$$-\frac{1}{2}(c_t - c) - \frac{1}{2}\delta \mathrm{E}[(C_{t+1} - c)^2]$$

s.t. $C_{t+1} = W_{t+1}$, $W_{t+1} = R_{W,t+1}(e_t - c_t)$, and $R_{W,t+1} = w_i R_{i,t+1}$.

$$\begin{aligned} M_{t+1} &= \delta \frac{C_{t+1} - c}{c_t - c} = \delta \frac{R_{W,t+1}(e_t - c_t) - c}{c_t - c} \\ &= -\frac{\delta c}{c_t - c} + \frac{\delta(e_t - c_t)}{c_t - c} R_{W,t+1} = a_t + b_t R_{W,t+1} \end{aligned}$$

Now, use the last result:

$$\mathrm{E}[R_{i,t+1}] = \gamma_t + \beta_{i,t}\lambda_t$$

where $\beta_{i,t} = (\mathrm{E}_t[R_{W,t+1} R_{W,t+1}])^{-1} \mathrm{E}_t[R_{W,t+1} R_{i,t+1}]$. How can we find λ?

The market risk premium

$\mathrm{E}_t[R_{W,t+1}]$ is determined by:

$$\mathrm{E}[R_{W,t+1}] = \gamma_t + 1\lambda_t$$

Thus:

$$\mathrm{E}[R_{i,t+1}] = \gamma_t + \beta_{i,t}(\mathrm{E}[R_{W,t+1}] - \gamma_t)$$

where γ_t is the expected return of a zero-beta asset.
If a risk-free asset exists, $\gamma_t = r_{f,t}$.

Rubinstein's (1976) log utility model

Define the "wealth portfolio" W as a claim to all future consumption. What is the price of this portfolio if the RA's utility from consumption c is $\ln[c]$?

$$M_{t+\tau} = \delta \frac{u'[C_{t+\tau}]}{u'[c_t]}$$

$$p_{W,t} = \mathrm{E}_t \sum_{\tau=0}^{\infty} M_\tau C_{t+\tau} = \mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau \frac{c_t}{C_{t+\tau}} C_{t+\tau} = \frac{\delta}{1-\delta} c_t$$

What is the return on the wealth portfolio?

$$R_{W,t+1} = \frac{p_{W,t+1} + C_{t+1}}{p_{W,t}} = \frac{\left(\frac{\delta}{1-\delta} + 1\right) C_{t+1}}{\frac{\delta}{1-\delta} c_t} = \frac{1}{\delta} \frac{C_{t+1}}{c_t} = \frac{1}{M_{t+1}}$$

Discussion

So, $M_{t+1} = 1/R_{W,t+1}$. Why do we get this result?

Because the effects of news of higher future consumption $C_{t+\tau}$ on the value of the wealth portfolio are offset by changes in the discount factor $M_{t+\tau}$.

How about a factor representation?

- The discrete time log utility model can only represented as $M_{t+1} = a_t + b_t R_{W,t+1}$ by means of a Taylor approximation.
- An exact linearization is often possible in continuous time since diffusion processes are locally normally distributed - we will see this when we talk about the ICAPM.
- Or, we assume normally distributed returns and factors...

Stein's lemma

Stein's lemma: If F and R are bivariate normal, ψ is a differentiable function and $\mathrm{E}[|\psi'[F]|] < \infty$, then

$$\mathrm{Cov}[\psi[F], R] = \mathrm{E}[\psi'[F]]\mathrm{Cov}[F, R]$$

Factor models for normally distributed returns

Suppose that $M_{t+1} = \psi[F_{t+1}]$. Then, for $X_{t+1} = p_t R_{t+1}$:

$$
\begin{aligned}
p_t &= \mathrm{E}_t[M_{t+1}X_{t+1}] = \mathrm{E}_t[\psi[F_{t+1}]X_{t+1}] \\
&= \mathrm{E}_t[\psi[F_{t+1}]]\mathrm{E}_t[X_{t+1}] + \mathrm{Cov}_t[\psi[F_{t+1}]X_{t+1}] \\
&= \mathrm{E}_t[\psi[F_{t+1}]]\mathrm{E}_t[X_{t+1}] + \mathrm{E}[\psi'[F_{t+1}]]\mathrm{Cov}[F_{t+1}, X_{t+1}] \\
&= \mathrm{E}_t[(\mathrm{E}_t[\psi[F_{t+1}]] + \mathrm{E}[\psi'[F_{t+1}]](F_{t+1}, -\mathrm{E}_t[F_{t+1}])X_{t+1}] \\
&= \mathrm{E}_t[(\mathrm{E}_t[\psi[F_{t+1}]] - \mathrm{E}[\psi'[F_{t+1}]]\mathrm{E}_t[F_{t+1}] + \mathrm{E}[\psi'[F_{t+1}]]F_{t+1})X_{t+1}] \\
1 &= \mathrm{E}_t\left[(\mathrm{E}_t[\psi[F_{t+1}]] - \mathrm{E}[\psi'[F_{t+1}]]\mathrm{E}_t[F_{t+1}] + \mathrm{E}[\psi'[F_{t+1}]]F_{t+1})\frac{X_{t+1}}{p_t}\right]
\end{aligned}
$$

i.e.

$$
\begin{aligned}
M_{t+1} &= \mathrm{E}_t[\psi[F_{t+1}]] - \mathrm{E}[\psi'[F_{t+1}]]\mathrm{E}_t[F_{t+1}] + \mathrm{E}[\psi'[F_{t+1}]]F_{t+1} \\
&= a_t + b_t F_{t+1}
\end{aligned}
$$

for any differentiable function ψ.

Merton's (1973) Intertemporal CAPM

Investors maximize:

$$V[e_t, \theta_t] = \max_{c_t, w_t} u[c_t] + \delta \mathrm{E}_t[V[W_{t+1}, \theta_t]]$$

where $W_{t+1} = R_{W,t+1}(e_t - c_t)$, $R_{W,t+1} = w_t R_{t+1}$ and θ_t are state variables that may describe relative price changes, changes in the investment opportunity set, etc.

We will consider a continuous time model. Recall:

$$\mathrm{E}_t\left[\frac{dp_{j,t}}{p_{j,t}}\right] + \frac{d_{j,t}}{p_{j,t}}dt = r_{f,t}dt - \mathrm{E}_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dp_{j,t}}{p_{j,t}}\right]$$

Assume that

$$\Lambda_t = \exp[-\delta t] V_W[W_t, \theta_t].$$

$$\frac{d\Lambda_t}{\Lambda_t} = -\delta dt + \frac{W_t V_{WW}[W_t, \theta_t]}{V_W[W_t, \theta_t]}\frac{dW}{W} + \frac{V_{W\theta}[W_t, \theta_t]}{V_W[W_t, \theta_t]} d\theta_t$$

A factor model

Let:

$$\mathrm{RR}_t = -\frac{W_t V_{WW}[W_t, \theta_t]}{V_W[W_t, \theta_t]}.$$

Then,

$$\begin{aligned}
\mathrm{E}_t\left[\frac{dp_{j,t}}{p_{j,t}}\right] + \frac{d_{j,t}}{p_{j,t}}dt &= r_{f,t}dt - \mathrm{E}_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dp_{j,t}}{p_{j,t}}\right] \\
&= r_{f,t}dt + \mathrm{RR}_t\mathrm{E}_t\left[\frac{dW_t}{W_t}\frac{dp_{j,t}}{p_{j,t}}\right] - \frac{V_{W\theta}[W_t, \theta_t]}{V_W[W_t, \theta_t]}\mathrm{E}_t\left[d\theta_t\frac{dp_t}{p_t}\right] \\
\mathrm{E}_t[R_{j,t+1}] &\approx r_{f,t} + \mathrm{RR}_t\mathrm{Cov}_t\left[R_{W,t+1}, R_{j,t+1}\right] + \lambda_t\mathrm{Cov}_t[\theta_{t+1}, R_{j,t+1}]
\end{aligned}$$

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Conditional and unconditional models

Recall:

$$p_{j,t} = \mathrm{E}_t[M_{t+1}X_{j,t+1}]$$

We can "condition down":

$$\mathrm{E}[[p_{j,t}] = \mathrm{E}[M_{t+1}X_{j,t+1}]$$

since your forecast of your forecast at t is your current forecast. Does the same work for factor models? Suppose

$$1 = \mathrm{E}_t[(a_t + b_t F_{t+1})R_{j,t+1}]$$

$$1 = \mathrm{E}_t[a_t]\mathrm{E}_t[R_{j,t+1}] + \mathrm{E}_t[b_t]\mathrm{E}_t[F_{t+1}R_{j,t+1}] + \mathrm{Cov}_t[a_t, R_{j,t+1}] + \mathrm{Cov}_t[b_t, F_{t+1}R_j$$

so, we need that $\mathrm{Cov}_t[a_t, R_{j,t+1}] = \mathrm{Cov}_t[b_t, F_{t+1}R_{j,t+1}] = 0$ for

$$1 = \mathrm{E}_t[(\mathrm{E}_t[a_t] + \mathrm{E}_t[b_t]F_{t+1})R_{j,t+1}]$$

Implications

A model that implies a conditional linear factor model with respect to a particular information set that cannot be observed. As a consequence, such models cannot be tested.

- Examples: the two period quadratic utility model, models based on normally distributed returns (Stein's lemma),...
- Exception: models like the log-utility model ($M = 1/R_W$) can be tested using GMM.

A partial solution: we could try to model the variation in a_t and b_t in $M_{t+1} = a_t + b_t F_{t+1}$:

$$a_t = a\theta_t \text{ and } b_t = b\theta_t$$

$$M_{t+1} = a\theta_t + b\theta_t F_{t+1}$$

is a model with factors $\theta_t, F_{t+1}, \theta_t F_{t+1}$. The variables z_t are referred to as "instruments".

This class

A statistical model

Suppose that asset returns can be described by the following "factor decomposition":

$$R_j = a_j + \beta_j F + \epsilon j$$

$$R_j = \mathrm{E}[R_j] + \beta_j (F - \mathrm{E}[F]) + \epsilon j$$

where $\mathrm{E}[\epsilon_j \epsilon_k] = 0$. (This restriction is the real "content" of the model.)

No arbitrage

Let's assume that we can set up well-diversified portfolios s.t.

$$R_P = wR \approx \mathrm{E}[wR] + \beta_P F$$

where w is the vector of portfolio weights and R is a vector of asset returns. No arbitrage requires that:

$$\begin{aligned}
1 &= \mathrm{E}[MR_P] \\
0 &= \mathrm{E}[M(R_P - r_f)] \\
0 &= \mathrm{E}[M(\mathrm{E}[R_P] + \beta_P F - r_f)] \\
\mathrm{E}[M](\mathrm{E}[R_P] - r_f) &= \beta_P(\mathrm{E}[MF]) \\
\mathrm{E}[R_P] - r_f &= \beta_P \frac{\mathrm{E}[MF]}{\mathrm{E}[M]} \\
\mathrm{E}[R_P] - r_f &= \beta_P \lambda \\
w\mathrm{E}[R] - r_f &= w\beta\lambda \\
\mathrm{E}[R_j] - r_f &= \beta_j \lambda
\end{aligned}$$

Factor risk premia

Let's construct a portfolio Q_i with a unit beta vector β_{Q_i} (i.e. a vector the ith component of which equals one while all other components are zero). Line 6 of our last derivation implies:

$$\mathrm{E}[R_{Q_i}] - r_f = \lambda_i$$

Thus,

$$\mathrm{E}[R_j] - r_f = \sum_i \beta_{j,i}(\mathrm{E}[R_{Q_i}] - r_f)$$

One open question remains: what are the factors?