Finance: Lecture 5 - Efficient markets
Chapters 2 and 3 of Ross (2005)

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This class

1. Foundations
2. Bounds on the pricing kernel
3. Market efficiency

Complete vs. incomplete markets

Recall:

- The pricing kernel is unique in a complete market, but not in an incomplete market.
- In an incomplete market, we can think about the pricing kernel in models such as the CCAPM. We know that the market must use a kernel in the set

$$\{m \mid \forall x, \mathrm{E}[mx] = 0\} \cap \mathbb{R}_+$$

  where x denotes an excess return (return net of the risk-free rate) of a marketed asset.

Incomplete vs. complete market portfolio choice

Lets compare investor behavior:

- Incomplete market: the investor solves

$$\max_{\alpha} \mathrm{E}u[e((1+r)+\alpha x)]$$

with FOC: $\mathrm{E}[u'[z]x]=0$ where $z=eR_\alpha$ for $R_\alpha=(1+r)+\alpha x)$. Solution: $z^*=eR_{\alpha^*}$.

- Complete market: the investor solves

$$\max_{z} \mathrm{E}u[z] \text{ s.t. } \mathrm{E}[mz]=e$$

with FOC: $u'[z]-\lambda m=0$ where λ is the shadow price of e. The solution will be the same as that of the first problem if:

$$m=\frac{u'[z^*]}{\lambda}=\mathrm{E}[m]\frac{u'[z^*]}{\mathrm{E}[u'[z^*]]}.$$

This is indeed a pricing kernel: $\mathrm{E}[mx]=(1/\lambda)\mathrm{E}[u'[z^*]x]=0$ by the FOC of the first problem.

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Digression

The case with complete markets can be seen as a standard problem in consumer theory:

- Indirect utility function: $V[m, e] = \max_{z|\mathrm{E}[mz]=e} \mathrm{E}[u[z]]$
- Expenditure function: $C[m, u] = \min_{z|\mathrm{E}[u[z]]\geq u} \mathrm{E}[mz]$

A standard result in microeconomics: C is concave in m.

Comparing pricing kernels

Let's compare two pricing kernels: $\hat{m} = m + \epsilon$ where $\mathrm{E}[\epsilon|m] = 0$. With complete markets, which kernel allows an investor to obtain a given utility level u with a smaller endowment?

$$C[m, u] = \mathrm{E}[mz^*_m] = \mathrm{E}[(m+\epsilon)z^*_m] \geq \mathrm{E}[\hat{m}z^*_{\hat{m}}] = C[\hat{m}, u].$$

Now, lets consider an incomplete market. We saw that we can find a pricing kernel that "supports the investor's optimal choice, z^*:

$$m = u'[z^*]/\lambda$$

This kernel must be sufficiently "variable"...

Example: Bansal and Lehmann (1997)

Assume logarithmic utility: $u'[z] = 1/z$.

- Incomplete market: the investor holds a "growth-optimal" portfolio with payoff $z^* = eR_{\alpha^*}$ for an indirect utility of

$$\mathrm{E}[u[eR_{\alpha^*}]] = \mathrm{E}[\ln[eR_{\alpha^*}]] = \ln[e] + \mathrm{E}[\ln[R_{\alpha^*}]]$$

- This choice is supported by the pricing kernel $m = 1/(z^*\lambda) = e/z^*$ (since $mz^* = e$ (budget constraint) holds if $\lambda = 1/e$). In a complete market, the pricing kernel would give the investor an indirect utility of

$$V[e/z^*, e] = \mathrm{E}[u[z^*]] = \mathrm{E}[\ln[z^*]] = \mathrm{E}[\ln[e/m]] = \ln e - \mathrm{E}[\ln[m]]$$

  The investor's indirect utility must be higher if the market is complete:

$$V[e/z^*, e] = \ln e - \mathrm{E}[\ln[m]] \geq V_{z^*} = \ln[e] + \mathrm{E}[\ln[R_{\alpha^*}]]$$

Thus:

$$\mathrm{E}[\ln[m]] \leq -\mathrm{E}[\ln[R_{\alpha^*}]]$$

More generally...

More generally, each choice of utility function leads to a bound on the pricing kernel:

- Quadratic utility: Hansen-Jagannathan (1991) bound
- Logarithmic utility: Bansal-Lehmann (1997) bound
- Exponential utility: Snow (1991) entropy bound

"Comparative statics": the more risk-averse an investor, the "riskier" the pricing kernel consistent with the investor achieving a given level of indirect utility with a given endowment (since we have seen that riskier kernels make it easier to reach a given level of utility).

An upper bound...

Suppose that we know that an investor's utility function u satisfies:

$$v = G[u]$$

where v is another utility function and G is a strictly concave/increasing function. Then, we can use the result (see Ross (2005), page 33.):

$$\mathrm{E}[m_u^2] \leq \mathrm{E}[m_v^2]$$

Example

Suppose that v is CRRA with $\gamma \neq 1$:

$$
\begin{aligned}
m_v &= \mathrm{E} m_v \frac{u'[z]}{E[u'[z]]} = \mathrm{E} m_v \frac{z^{-\gamma}}{\mathrm{E}[z^{-\gamma}]} \\
\mathrm{E}[m_v^2] &= (\mathrm{E} m_v)^2 \frac{\mathrm{E}[z^{-2\gamma}]}{(E[z^{-\gamma}])^2}
\end{aligned}
$$

Suppose $\ln z \sim N[\mu, \sigma^2]$. Then $\mathrm{E}[z^k] = \exp[k\mu + k^2\sigma^2/2]$ for any k, and

$$\mathrm{E}[m_v^2] = (\mathrm{E} m_v)^2 \exp[\gamma^2 \sigma^2]$$

We obtain a bound on the variance of the pricing kernel of an agent with utility u: since $\mathrm{E}[m_u^2] \leq \mathrm{E}[m_v^2]$ and the mean of the pricing kernel must equal $1/(1 + r_f)$ (under both u and v)

$$\mathrm{E}[m_u^2] \leq \mathrm{E}[m_v^2] \Rightarrow \mathrm{Var}[m_u^2] \leq \mathrm{Var}[m_v^2]$$

where

$$\mathrm{Var}[m_v^2] = \mathrm{E}[m_v^2] - (\mathrm{E} m_v)^2 = (\mathrm{E} m_v)^2(\exp[\gamma^2\sigma^2] - 1) \approx (\mathrm{E} m_v)^2 \gamma^2 \sigma^2$$

Summary: bounds on the pricing kernel

We had the Hansen-Jagannathan bound:

$$\mathrm{E}[M]\frac{\mathrm{E}[R_j]}{\sigma_j} \leq \sigma_M$$

Now, we have the following bound:

$$\mathrm{E}[m]^2 = \mathsf{Var}[m]^2 + (\mathrm{E}m)^2 \approx \mathsf{Var}[m]^2 \leq (\mathrm{E}m_v)^2\gamma^2\sigma^2_{R_{eff}} = (\mathrm{E}m)^2\gamma^2\sigma^2_{eff}.$$

i.e.

$$\sigma_M \leq \mathrm{E}[M]\gamma\sigma_{eff}$$

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Definition

Let I_t denote the "market information set", defined by

$$p_t = \frac{1}{1+r_f}\mathrm{E}^*[z_{t+1}|I_t]$$

Fama (1970): the market is...

- ... weak form efficient if $I_t = \{..., p_{t-2}, p_{t-1}, p_t\}$
- ... semi-strong form efficient if I_t includes any information that is publicly available at time t.
- ... strong form efficient if I_t contains all information available to any living soul.

Implications of market efficiency for investing

An investment strategy based on an information set $A_t \subset I_t$ cannot generate value.
Proof: let z denote the vector of asset payoffs, and let α denote an investment strategy conditional on A_t. Then:

$$\begin{aligned}
\frac{1}{1+r_f}\mathrm{E}^*[\alpha z_{t+1}|A_t] &= \frac{1}{1+r_f}\mathrm{E}^*[\mathrm{E}^*[alphaz_{t+1}|S_t]|A_t] \\
&= \frac{1}{1+r_f}\mathrm{E}^*[\alpha\mathrm{E}^*[z_{t+1}|S_t]|A_t] \\
&= \frac{1}{1+r_f}\mathrm{E}^*[\alpha(1+r_f)p_t|A_t] = p_t
\end{aligned}$$

Corollary: an investment strategy conditioned on $A_t \subset I_t$ must yield a risk-adjusted expected return of r_f. (And, of course, the same is true unconditionally.)

Implications of market efficiency for returns

Weak form efficiency implies that returns are uncorrelated with any linear combination of past returns if correlations are computed using risk-neutral probabilities.

- Returns may be serially correlated because risk premia may be correlated with past prices. Let $L[p_{-t}]$ denote a linear combination of past returns:

$$\begin{aligned}
\mathrm{Cov}[R_t, L[p_{-t}]] &= \mathrm{E}[(R_t - \mathrm{E}[R_t])L[p_{-t}]] \\
&= \mathrm{E}[\mathrm{E}[R_t - \mathrm{E}[R_t]|L[p_{-t}]]L[p_{-t}]] \\
&= \mathrm{E}[\mathrm{E}[R_t - (r_{f,t} + \mathrm{E}[\lambda_t])|L[p_{-t}]]L[p_{-t}]] \\
&= \mathrm{E}[(\mathrm{E}[\lambda_t|L[p_{-t}]] - \mathrm{E}[\lambda_t])L[p_{-t}]] \\
&= \mathrm{Cov}[\lambda_t, L[p_{-t}]]
\end{aligned}$$

- Empirical tests of weak form efficiency require that we use risk-adjusted returns. Such tests are joint tests of both market efficiency and an asset pricing model.

Testing market efficiency

How compromised are tests of market efficiency by our not knowing risk premia? If returns turn out to be predictability, how predictable must they be before we can reject market efficiency?

- Suppose that the excess return on an asset is:

$$X_t = \mu[I_t] + \epsilon_t$$

  where $\mu[I_t] = \mathrm{E}[X_t|I_t]$, $\sigma_x^2 = \sigma_\mu^2 + \sigma_\epsilon^2$, and $\sigma_\mu^2 = \mathrm{E}[(\mu[I_t] - \mathrm{E}[\mu[I_t]])^2]$.
- The Hansen-Jagannathan bound

$$\mathrm{E}[X_t] = \mu[I_t] \leq \sigma_x \frac{\sigma_M}{\mathrm{E}[M]}$$

  implies that $\sigma_\mu^2 = \mathrm{E}[(\mu[I_t] - \mathrm{E}[\mu[I_t]])^2] \leq \mathrm{E}[(\mu[I_t])^2 \leq \sigma_x^2 \frac{\sigma_M^2}{(\mathrm{E}[M])^2}$.

We can use this result to test market efficiency...

Testing market efficiency

Suppose that we know that the variance of the pricing kernel is at most σ_M^2. Then, market efficiency implies that a regression of an excess return X_t on the information set I_t should have an $R^2 \leq (1+r_f)^2\sigma_M^2$.
Proof:

$$\begin{aligned} R^2 &= \frac{\sigma_\mu^2}{\sigma_x^2} \\ \sigma_x^2 R^2 &\leq \sigma_m u^2 \\ \sigma_x^2 R^2 &\leq \sigma_x^2 \frac{\sigma_M^2}{(\mathrm{E}[M])^2} \\ R^2 &\leq (1+r_f)^2\sigma_M^2 \end{aligned}$$

Back-of-the-envelope calculations

Recall:

$$\sigma_M \leq \mathrm{E}[M]\gamma\sigma_{eff} = \frac{1}{1+r_f}\gamma\sigma_{eff}$$

Let's use the S&P500 as a proxy: $\sigma_{eff} = 0.2 p.a.$. $r_f = 0.035 p.a.$. Assume $\gamma = 5$ as an upper bound on risk aversion.

- For daily data, we obtain that $\sigma_M \leq 0.05$. If $\sigma_M = 0.05$, we would obtain $R^2 \leq 0.0025$ as an upper bound on the R^2 of, say, serial return correlation.
- For weekly data, we obtain $\sigma_M \leq 0.138$ and $R^2 \leq 0.019$.