Finance: Lecture 4 - No Arbitrage Pricing
Chapters 10 -12 of DD
Chapter 1 of Ross (2005)

Prof. Alex Stomper

MIT Sloan, IHS & VGSF

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This class

No arbitrage

We use a model with two points in time, s possible states $(\theta_1, ..., \theta_s)$ and n traded assets. $p = (p_1, ..., p_N)$ is the price vector, Z is the payoff matrix (rows are states, columns are assets).

- A portfolio $\eta = (\eta_1, ...\eta_n)$ costs $p\eta$. The payoff vector is $Z\eta$.
- An arbitrage opportunity is a portfolio η that (i) requires no investment, (ii) will not yield a loss, and (iii) may return a strictly positive gain.

$$\begin{aligned} p\eta &\leq 0 \\ Z\eta &> 0 \end{aligned}$$

where $x > y$ means that all (some) components of a vector x are greater or equal to (strictly greater than) the corresponding components of a vector y.

The fundamental theorem

The following statements are equivalent:

(i) There do not exist any arbitrage opportunities.

(ii) There exists a positive linear pricing rule q that prices all assets: $p = qZ$, where all elements of q are strictly positive.

(iii) Each agent has a finite optimal demand for all assets.

Proof:

(ii)$\rightarrow$(i): let η be an arbitrage opportunity. Then, $0 \geq p\eta = (qZ)\eta = q(Z\eta)$. Since q is positive, we obtain a contradiction in that $Z\eta < 0$.

Proof: (i)→(ii) (sketch)

Absence of arbitrage implies that the set of feasible cost/payoff combinations $\{x | \exists \eta, x = (p\eta, Z\eta)\}$ intersects with $\mathbb{R}_+ \times \mathbb{R}_+^S$ at zero.

Since $\mathbb{R}_+ \times \mathbb{R}_+^S$ is a cone, use a special version of the separating hyperplane theorem: there exists a separating axis s.t. the projection of *any* point in the set of feasible cost-payoff combinations onto the axis is *strictly* smaller than that of any point in the interior of $\mathbb{R}_+ \times \mathbb{R}_+^S$.

> For any such interior point $x > 0$ and any vector $(1, q)$ representing such a projection, we thus have $(1, q)x > 0$ (since zero is the projection of the zero cost/payoff combination).
>
> Since $x > 0$, $q > 0$.

Moreover, for any $x \in \{x | \exists \eta, x = (p\eta, Z\eta)\}$: $(1, q)x = (1, q)A\eta \leq 0$ and (since $-\eta$ is also feasible) $(1, q)A\eta = 0$, where

$$A = \begin{pmatrix} -p \\ Z \end{pmatrix}$$

Uniqueness of the pricing rule

If the market is complete, Z has full row-rank, and $p = qZ$ has a unique solution:

$$q = pZ^{-1}$$

In an incomplete market, the securities' payoff vectors don't "span" the state space, and the pricing rule is not unique.

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The representation theorem

The following statements are equivalent:

(i) There exists a positive linear pricing rule.

(ii) The martingale property: there exist martingale or risk-neutral probabilities (or a density) and an associated riskless rate.

(iii) There exists a positive pricing kernel or state price density.

We will see the requisite definitions as we go along...

Risk-neutral probabilities / martingale probabilities

We start by valuing a riskless payoff.

$$\frac{1}{1+r_{f,t+1}} = q1 = \sum q_i$$

Define: risk neutral probabilities as normalized prices:

$$\pi^* = \frac{q}{\sum q_i}$$

Now we value a payoff X with a vector of realizations x:

$$p = qx = \sum q_i \pi^* x = \frac{1}{1+r_{f,t+1}} \pi^* x = \frac{1}{1+r_{f,t+1}} \mathrm{E}^* x$$

where i is the index for states.

Pricing kernel / state price density

A pricing kernel has been defined by:

$$p = \mathrm{E}MX = \pi m x$$

where M denotes the pricing kernel (with a vector of realizations m) and π denotes the probability vector.
Since $p = qx$, we can define the pricing kernel as a vector with components

$$m_i = \frac{q_i}{\pi_i}$$

Will the components of the pricing kernel sum to one?

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Measuring the state-price density

What is the value of a security that yields a unit payoff in state *i* and zero in any other state?

Suppose that the state space consists of 6 states in which the payoff of a stock is $\{8, 9, ..., 13\}$. Here is a table of call prices of calls on the stock with different strike prices.

strike price	7	8	9	10	11	12	13
call price	3.22	2.32	1.54	0.94	0.53	0.28	0.14

Measuring the state-price density: ctd.

Construct butterfly spreads:

strike	call	bfly	Payoff if stock price is...						1st diff	2nd diff
			8	9	10	11	12	13		
7	3.22									
									-0.9	
8	2.32									0.12
									-0.78	
9	1.54	1.54	0	0	1	2	3	4		0.18
									-0.60	
10	0.94	-1.88	0	0	0	-2	-4	-6		**0.19**
									-0.41	
11	0.53	0.53	0	0	0	0	1	2		0.16
									-0.25	
12	0.28									0.11
									-0.14	
13	0.14									
		0.19	0	0	**1**	0	0	0		

Option pricing with the binomial model

We want to value an option on a stock that is currently worth p_S and may be worth either up_S or dp_S when the option expires. How can we value the option?

- There are two states and three assets: the stock, the riskfree asset, and the call option. Local market completeness.
- What is the pricing rule? Solve:

$$\begin{aligned} p_S &= q_u u p_S + q_d d p_S \\ 1 &= q_u(1+r_f) + q_d(1+r_f) \end{aligned}$$

i.e.

$$q_u = \frac{1+r_f-d}{(1+r_f)(u-d)} \text{ and } q_d = \frac{u-(1+r_f)}{(1+r_f)(u-d)}$$

Why is it true that $q = (q_u, q_d) > 0$?

Option pricing continued

We had:

$$q_u = \frac{1 + r_f - d}{(1 + r_f)(u - d)} \text{ and } q_d = \frac{u - (1 + r_f)}{(1 + r_f)(u - d)}$$

Both are positive since no arbitrage requires that $u > 1 + r_f > d$.

- Risk-neutral probabilities:

$$\pi^* = \pi_u^* + \pi_d^* = \frac{q_u}{q_u + q_d} + \frac{q_d}{q_u + q_d} = \frac{1 + r_f - d}{(u - d)} + \frac{u - (1 + r_f)}{(u - d)} = 1$$

- Pricing kernel:

$$m_u = \frac{q_u}{\pi_u} = \frac{1 + r_f - d}{\pi_u (1 + r_f)(u - d)} \text{ and } m_d = \frac{q_d}{\pi_d} = \frac{u - (1 + r_f)}{\pi_d (1 + r_f)(u - d)}$$

- Let the option payoff be either x_u or x_d. The option price is:

$$p = q_u x_u + q_d x_d = \frac{\pi_u^* x_u + \pi_d^* x_d}{1 + r_f} = \pi_u m_u x_u + \pi_d m_d x_d$$