Finance: Lecture 3 - The CCAPM
Chapters 8 and 9 of DD, Chapter 1 of Cochrane (2001)

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This class

1. An Arrow-Debreu Economy
2. The Consumption CAPM
3. Continuous time
4. Focus topic: habit formation in the CCAPM

An Arrow-Debreu Economy - Section 8.2

Elements:

- Dates: t and $t+1$.
- Goods: one perishable (i.e. non-storable) consumption good (fruit, according to Lucas (1987)).
- States of nature characterized by the output Y_{t+1} of the economy (the fruit tree) at date $t+1$:

$$G[Y_{t+1}|Y_t] = P[Y_{t+1} \leq Y^j | Y_t = Y^i].$$

- A representative agent maximizing:

$$\max_{z_{t+1}} \mathrm{E}\left[\sum \delta^t u[C_t]\right]$$

$$c_t + p_t z_{t+1} \leq z_t y_t + p_t z_t$$

$$z_t \leq 1$$

where p_t is the period t price of fruit, and z_t is the agent's share of the fruit tree at the beginning of period t.

Comments

1. The representative agent construction leads to AD prices that are the same as in an economy in which

- all agents use the same discount factors and
- all agents' can be represented by CRRA or CARA.

If these conditions are violated, we need to know the agents' endowments in order to construct the representative agent.

2. Rational expectations: the representative agent knows the structure of the economy.

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Solving for the equilibrium

For all t, z_{t+1} solves:

$$u'[c_1]p_t = \delta \mathrm{E}_t u'[C_{t+1}](P_{t+1} + Y_{t+1}).$$

Moreover, $z_t = z_{t+1} = ... = 1$ and $c_t = Y_t$.
The equilibrium price must therefore satisfy:

$$u'[y_t]p_t = \delta \mathrm{E}_t u'[Y_{t+1}](P_{t+1} + Y_{t+1}).$$

Since

$$u'[Y_{t+1}]P_{t+1} = \delta \mathrm{E}_{t+1} u'[Y_{t+2}](P_{t+2} + Y_{t+2}),$$

we obtain

$$p_t = \delta \mathrm{E}_t \frac{u'[Y_{t+1}]}{u'[y_t]} Y_{t+1} + \delta^2 \mathrm{E}_t \mathrm{E}_{t+1} \frac{u'[Y_{t+2}]}{u'[y_t]}(P_{t+2} + Y_{t+2})$$

Use the same trick over and over:

$$p_t = \mathrm{E}_t \sum \delta^\tau \left(\frac{u'[Y_{t+\tau}]}{u'[y_t]} Y_{t+\tau} \right).$$

Interpretation - Section 9.3

Define (gross returns, rather than net returns as in DD):

$$r_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t}$$

Then:

$$u'[y_t]p_t = \delta \mathrm{E}_t u'[Y_{t+1}](P_{t+1} + Y_{t+1}).$$

implies

$$1 = \mathrm{E}_t m_{t+1}[S_{t=1}]R_{t+1}[S_{t+1}].$$

where $m_{t+1}[S_{t+1}]$ is a (state S_{t+1} dependent) discount factor, referred to as the *pricing kernel*:

$$m_{t+1}[S_{t+1}] = \delta \frac{u'[y_{t+1}[S_{t+1}]]}{u'[Y_t]} = \delta \frac{u'[c_{t+1}[S_{t+1}]]}{u'[C_t]}.$$

Another way of writing the pricing kernel

Original definition:

$$M_{t+1} = \delta \frac{u'[C_{t+1}]}{u'[c_t]}$$

Since

$$\mathrm{E}_t M_{t+1} = \delta \frac{\mathrm{E}[u'[C_{t+1}]]}{u'[c_t]}$$

we can define the pricing kernel as follows:

$$\begin{aligned} M_{t+1} &= \delta \frac{u'[C_{t+1}]}{u'[c_t]} \\ &= \mathrm{E}_t M_{t+1} \frac{u'[c_t]}{\mathrm{E}[u'[C_{t+1}]]} \frac{u'[C_{t+1}]}{u'[c_t]} \\ &= \mathrm{E}_t M_{t+1} \frac{u'[C_{t+1}]}{\mathrm{E}[u'[C_{t+1}]]}. \end{aligned}$$

A riskless asset

Suppose that we create a risk-free asset (in zero net supply). Per unit of risk-free payoff, the price is:

$$u'[c_t]q_t^b = \delta \mathrm{E}_t u'[C_{t+1}].$$

such that the risk-free rate (again, gross!) is:

$$\frac{1}{r_{f,t+1}} = q_t^b = \mathrm{E}_t m_{t+1}[S_{t=1}].$$

Example

Let's specify power utility $u'[c] = c^{-\gamma}$ and "turn off" uncertainty:

$$\mathrm{E}_t m_{t+1}[S_{t=1}] = \delta \left(\frac{c_t}{c_t + 1} \right)^{\gamma}$$

$$r_{f,t+1} = \frac{1}{\delta} \left(\frac{c_{t+1}}{c_t} \right)^{\gamma}$$

- Real interest rates are high when the RA is impatient, and when consumption growth is high.
- The sensitivity of real interest rates wrt consumption growth depends on γ: the higher γ, the stronger the RA's preference for consumption smoothing, the higher the interest rate it takes to induce consumption growth.

The Consumption CAPM

We split the fruit tree into stocks indexed by j with dividends $d_{j,t+1}$.

$$\begin{aligned} 1 &= \mathrm{E}_t m_{t+1}[S_{t+1}] r_{j,t+1}[S_{t+1}] = \mathrm{E}_t M_{t+1} R_{j,t+1} \\ &= \mathrm{E}_t M_{t+1} \mathrm{E} R_{j,t+1} + \mathsf{Cov}_t \left[M_{t+1}, R_{j,t+1} \right] \\ &= \frac{\mathrm{E}_t R_{j,t+1}}{r_{f,t+1}} + \mathsf{Cov}_t \left[M_{t+1}, R_{j,t+1} \right] \end{aligned}$$

such that $\mathrm{E}_t R_{j,t+1} - r_{f,t+1} = -r_{f,t+1} \mathsf{Cov} \left[M_{t+1}, R_{j,t+1} \right]$. Rearranging and substituting for $r_{f,t+1} = 1/(\mathrm{E} M_{t+1})$ yields:

$$\mathrm{E}_t R_{j,t+1} - r_{f,t+1} = -\frac{\mathsf{Var}[M_{t+1}]}{\mathrm{E}_t M_{t+1}} \frac{\mathsf{Cov}_t \left[M_{t+1}, R_{j,t+1} \right]}{\mathsf{Var}[M_{t+1}]} = \lambda_{M,t} \beta_{j,M,t}.$$

Hansen-Jagannathan bounds - Section 9.6

We had:

$$\begin{aligned} 1 &= \mathrm{E}_t M_{t+1} \mathrm{E}_t R_{j,t+1} + \mathrm{Cov}_t[M_{t+1}, R_{j,t+1}] \\ &= \mathrm{E}_t M_{t+1} \mathrm{E}_t R_{j,t+1} + \rho_t[M_{t+1}, R_{j,t+1}]\sigma_{j,t}\sigma_{M,t} \end{aligned}$$

such that

$$\frac{1}{\mathrm{E}_t M_{t+1}} = \mathrm{E}_t R_{j,t+1} + \rho_t[M_{t+1}, R_{j,t+1}]\sigma_{j,t}\frac{\sigma_{M,t}}{\mathrm{E}_t M_{t+1}}.$$

Hansen-Jagannathan bounds:

$$\begin{aligned} |\mathrm{E}R_{j,t+1} - r_{f,t+1}| &\leq \sigma_{j,t}\frac{\sigma_{M,t}}{\mathrm{E}_t M_{t+1}} \\ \frac{|\mathrm{E}R_{j,t+1} - r_{f,t+1}|}{\sigma_{j,t}} &\leq \frac{\sigma_{M,t}}{\mathrm{E}_t M_{t+1}} \\ \frac{|\mathrm{E}R_{j,t+1} - r_{f,t+1}|}{\sigma_{j,t}} &\leq \frac{\lambda_{M,t}}{\sigma_{M,t}} \end{aligned}$$

Example: the equity premium puzzle - Section 9.5

Recall: across all assets j

$$\frac{\mathrm{E}R_{j,t+1} - r_{f,t+1}}{\sigma_{j,t}} \leq \frac{\lambda_{M,t}}{\sigma_{M,t}} = \frac{\sigma_{M,t}}{\mathrm{E}_t M_{t+1}}$$

In the power utility case, with lognormal consumption growth:

$$\frac{\mathrm{E}R_{j,t+1} - r_f}{\sigma_j} \leq \frac{\sigma_t[(C_{t+1}/c_1)^{-\gamma}]}{\mathrm{E}_t[(C_{t+1}/c_1)^{-\gamma}]}$$

Equity premium puzzle: based on historical data, the vola of consumption growth seems way too small to explain Sharpe ratios unless $\gamma \approx 50$.

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A diffusion model

Let d_t be the dividend yield of an asset at time t. Then, the instantaneous return on the asset is:

$$\frac{dp_t}{p_t} + \frac{d_t}{p_t} dt$$

for

$$\frac{dp_t}{p_t} = \mu[\cdot]dt + \sigma[\cdot]dz$$

where dz is standard normally distributed.
The CCAPM follows from maximizing

$$\mathrm{E}_t \int_{\tau=0}^{\infty} e^{-\delta\tau} u[c_\tau] d\tau$$

where the expectation is taken across "paths" of the diffusion process(es).

Derivation

FOCs:

$$\begin{aligned} p_t u'[c_t] &= \mathrm{E}_t \int_{\tau=0}^{\infty} e^{-\delta\tau} u'[c_{t+\tau}] d_{t+\tau} d\tau \\ p_t \Lambda_t &= \mathrm{E}_t \int_{\tau=0}^{\infty} \Lambda_{t+\tau} d_{t+\tau} d\tau, \end{aligned}$$

where $\Lambda_t = e^{-\delta t} u'[c_t]$. Also, at time $t + \Delta$:

$$p_{t+\Delta}\Lambda_{t+\Delta} = \mathrm{E}_{t+\Delta} \int_{\tau=0}^{\infty} \Lambda_{t+\Delta+\tau} d_{t+\Delta+\tau} d\tau$$

which must also be true in expectation as of time t. Subtract the last line from the next-to-last line to obtain, for small Δ:

$$\begin{aligned} p_t \Lambda_t &= \Lambda_t d_t \Delta + \mathrm{E}_t[\Lambda_{t+\Delta} p_{t+\Delta}] \\ &= \Lambda_t d_t \Delta + \mathrm{E}_t[p_t \Lambda_t + (\Lambda_{t+\Delta} p_{t+\Delta} - \Lambda_t p_t)] \\ 0 &= \Lambda_t d_t dt + \mathrm{E}_t[d(\Lambda_t p_t)] \end{aligned}$$

Interpretation

$$0 = \Lambda_t d_t dt + \mathrm{E}_t[d(\Lambda_t p_t)]$$

says that price follows a martingale if $d_t = 0$.
We use Ito's lemma: $d(\Lambda p) = d\Lambda p + \lambda dp + d\Lambda dp$
in order to obtain

$$0 = \frac{d_t}{p_t} dt + \mathrm{E}_t\left[\frac{d\Lambda_t}{\Lambda_t} + \frac{dp_t}{p_t} + \frac{d\Lambda_t}{\Lambda_t}\frac{dp_t}{p_t}\right]$$

Suppose $p_t = 1$, $dp_t = 0$, and $d_t = r_{f,t}$. Then, the equation reads:

$$0 = r_{f,t} dt + \mathrm{E}_t\left[\frac{d\Lambda_t}{\Lambda_t}\right]$$

We can therefore generally write:

$$\mathrm{E}_t\left[\frac{dp_t}{p_t}\right] + \frac{d_t}{p_t} dt = r_{f,t} dt - \mathrm{E}_t\left[\frac{d\Lambda_t}{\Lambda_t}\frac{dp_t}{p_t}\right]$$

Summary: CCAPM

The "mother of asset-pricing models" (Tyler Shumway):

$$
\begin{aligned}
1 &= \mathrm{E}_t[M_{t+1}R_{j,t+1}] \\
0 &= \mathrm{E}_t[M_{t+1}R_{j,t+1}] - 1 \\
&= \mathrm{E}_t[M_{t+1}(R_{j,t+1} - 1/\mathrm{E}_t M_{t+1})] \\
&= \mathrm{E}_t[M_{t+1}(R_{j,t+1} - r_{f,t+1})]
\end{aligned}
$$

Multiply both sides of $1 = \mathrm{E}_t[M_{t+1}R_{j,t+1}]$ by $p_{j,t}$:

$$p_{j,t} = \mathrm{E}_t[M_{t+1}X_{j,t+1}]$$

where $X_{j,t+1} = R_{j,t+1}p_{j,t} = D_{j,t+1} + P_{j,t+1}$.
Factor representation:

$$
\begin{aligned}
\mathrm{E}_t R_{j,t+1} - r_{f,t+1} &= -\frac{\mathrm{Var}_t[M_{t+1}]}{\mathrm{E}_t M_{t+1}} \frac{\mathrm{Cov}_t\left[M_{t+1}, R_{j,t+1}\right]}{\mathrm{Var}_t[M_{t+1}]} \\
&= \lambda_{M,t}\beta_{j,M,t}
\end{aligned}
$$

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Habit formation

Habit formation has a long history in the literature on consumption. For example, see Deaton's "Understanding consumption", 1992, Oxford University Press. A paper that analyzes habit formation in the CCAPM is: Cochrane and Campbell, 1999, "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior", Journal of Political Economy.

- Identical agents maximize the utility function:

$$\mathrm{E}_t \sum_t \delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

  X_t is the "habit" level. "Surplus consumption" $S_t = (C_t - X_t)/C_t$. What if $C_t < X_t$?

- R_A is now a function of C_t and X_t:

$$R_{A,t} = \frac{\gamma}{C_t - X_t} = C_t \frac{\gamma}{S_t}.$$

Habit accumulation

CC use an external habit specification: habit depends on the history of aggregate consumption C^a, rather than individual consumption:

- X_t is determined by:

$$X_t = C_t^a(1 - S_t^a)$$

  where S_t^a is a variable whose log $\ln S_t^a$ evolves according to a heteroskedastic AR(1) process:

$$\ln S_t^a = (1-\phi)\bar{s} + \phi \ln S_t^a + \lambda[\ln S_t^a](\ln \frac{C_{t+1}^a}{C_t^a} - g)$$

  where ϕ, g and $\bar{s}$ are parameters.

- Since all agents are identical: $C_t^a = C_t$, and

$$X_t = C_t(1 - \exp[(1-\phi)\bar{s} + \phi \ln S_t + \lambda[\ln S_t](\ln C_{t+1} - \ln C_t - g)].$$

Consumption growth

Consumption growth is an i.i.d. lognormal process:

$$\ln \frac{C_{t+1}}{C_t} = g + \nu_{t+1}$$

where $\nu_{t+1} \sim N[0, \sigma_\nu^2]$.
This consumption growth process can be interpreted either as an endowment process (the fruit tree) or as the consumption process of a "production" economy with a risk-free asset.

The pricing kernel

The derivative of the utility function wrt consumption C_t is:

$$u' = (C_t - X_t)^{-\gamma} = S_t^{-\gamma} C_t^{-\gamma}$$

The pricing kernel is:

$$\begin{aligned} M_{t+1} &= \delta \left(\frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma} \\ &= \delta(\exp[g])^{-\gamma} \exp[-\gamma((\phi - 1)(\ln S_t - \bar{s}) + (\lambda[\ln S_t] + 1)\nu_{t+1})] \end{aligned}$$

since

$$\begin{aligned} \frac{S_{t+1}}{S_t} &= \exp[\ln S_{t+1} - \ln S_t] = \exp[(\phi - 1)(\ln S_t - \bar{s}) + \lambda[\ln S_t]\nu_{t+1}] \\ \frac{C_{t+1}}{C_t} &= \exp[\ln C_{t+1} - \ln C_t] = \exp[g + \nu_{t+1}] \end{aligned}$$

The maximum Sharpe ratio

M_{t+1} is lognormal. For a lognormal random variable X:

$$\begin{aligned}\mathrm{E}X &= \exp[\mu + \sigma^2/2] \\ \mathrm{Var}X &= \exp[2\mu + \sigma^2](\exp[\sigma^2] - 1)\end{aligned}$$

such that

$$\frac{\sigma_X}{\mathrm{E}X} = \sqrt{\exp[\sigma^2] - 1}$$

and

$$\frac{\sigma_{M,t}}{\mathrm{E}_t M_{t+1}} = \sqrt{\exp[\gamma^2(\lambda[\ln S_t] + 1)^2 \sigma_\nu^2] - 1} \approx \gamma(\lambda[\ln S_t] + 1)\sigma_\nu$$

The Sharpe ratio depends on consumption growth risk and the sensitivity of the habit level wrt consumption growth shocks.

The riskfree rate

Recall: $r_{f,t+1} = 1/\mathrm{E}M_{t+1}$.

$$\begin{aligned}\frac{1}{\mathrm{E}_t M_{t+1}} &= 1/\exp[\ln\delta - \gamma g - \gamma(\phi - 1)(\ln S_t - \bar{s}) + \gamma^2(\lambda[\ln S_t] + 1)^2\sigma_\nu^2/2] \\ \ln r_{f,t+1} &= -\ln\delta + \gamma g - \gamma(1-\phi)(\ln S_t - \bar{s}) - \gamma^2(\lambda[\ln S_t] + 1)^2\sigma_\nu^2/2\end{aligned}$$

The interest rate...

- ...decreases in the surplus consumption ratio.
- ...decreases in the consumption growth risk.