Research Statement

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My research focuses primarily on interactions between several areas of mathematics, including algebraic geometry, commutative algebra, $\mathcal{D}$-modules and representation theory. An overarching theme in my work is the geometry of algebraic varieties under the presence of symmetries, that is, varieties equipped with actions of algebraic groups. We observe symmetries instinctively in nature as they guide us in recognizing distinguished objects such as cubes, spheres etc. Symmetries arise equally in many abstract problems; for instance, whenever we are working with some coordinates “up to a change of basis”, it is natural to carry over the action of the general linear group and use its representation theory for various computations. The varieties in my research that are endowed with actions of large groups include determinantal varieties, varieties of representations of quivers, nullcones, prehomogeneous vector spaces, Vinberg representations arising from gradings on Lie algebras (e.g. symmetric- and skew-symmetric matrices, space of binary cubic forms), and other spaces with finitely many orbits (e.g. toric and more generally spherical varieties).

I enjoy investigating subtle geometric properties and developing tools to calculate explicitly various invariants that are notoriously difficult to find. These include Bernstein-Sato polynomials, explicit structures of $\mathcal{D}$-modules such as local cohomology groups, mixed Hodge structures, Lyubeznik numbers, (minimal) generators and free resolutions of ideals, geometric properties as normality, Cohen-Macaulay, or rational singularities.

The theory of $\mathcal{D}$-modules arose as a bridge between various fundamental ideas coming from analysis, geometry, algebra and topology. One of the culminating results of the theory is the Riemann-Hilbert correspondence for (regular) holonomic $\mathcal{D}$-modules, a far-reaching generalization of Hilbert’s twenty-first problem, that has found many applications in modern mathematics. Furthermore, the theory has expanded into parts of applied mathematics. Recently, I have been investigating $\mathcal{D}$-modules appearing in statistics via the Holonomic Gradient Method that was initiated in [NNN+11]. In the paper [ALSS19], we developed this method for statistical inference from samples of rotations in 3-space, and extended the scope of the results to arbitrary compact Lie groups. Since $\mathcal{D}$-modules are becoming increasingly popular, software packages have been developed to handle various calculations based on non-commutative Gröbner basis methods. I am currently engaged in some of the computational aspects of the theory, developing some codes for the computer algebra system Macaulay2 such as computing the intersection cohomology groups of affine varieties.

Now I introduce some basics on quivers, as they appear in my work frequently. They arose originally from the study of finite dimensional algebras and their modules. A
quiver $Q$ is a finite oriented graph. A representation of $Q$ is an assignment of vector spaces to each vertex and linear maps to each arrow. A quiver $(Q, R)$ (with relations) is a quiver $Q$ together with a finite set of relations $R$; a relation is a linear combination of paths in $Q$. A representation $V$ of $(Q, R)$ is a representation of $Q$ such that the linear maps of $V$ satisfy the relations imposed by $R$.

In the following sections I present briefly the central topics of my research.

1 $\mathcal{D}$-modules and applications

1.1 Introduction

For simplicity, assume $X = \mathbb{C}^d$ is the complex affine space. Let $\mathcal{D}_X$ denote the ring of differential operators, or the Weyl algebra. It is generated as $\mathcal{D}_X = \mathbb{C}\langle x_1, \ldots, x_d, \partial_1, \ldots, \partial_d \rangle$, with relations $[\partial_i, x_j] = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta. A $\mathcal{D}$-module $M$ is just a module over the algebra $\mathcal{D}_X$. A $\mathcal{D}$-module is holonomic if it has “smallest” possible size, in which case it has finite length over $\mathcal{D}$.

Now assume that $X$ is equipped with the action of a connected algebraic group $G$. Differentiating the action of $G$ yields some vector fields on $X$, hence a map $\mathfrak{g} \to \mathcal{D}_X$, where $\mathfrak{g}$ is the Lie algebra of $G$. A $\mathcal{D}_X$-module $M$ is (strongly) equivariant if the action of $\mathfrak{g}$ on $M$ via the map $\mathfrak{g} \to \mathcal{D}_X$ can be integrated to an algebraic $G$-action on $M$.

Equivariant $\mathcal{D}_X$-modules arise naturally in many ways. For example, the coordinate ring $\mathbb{C}[X]$ or its localization at a semi-invariant polynomial, local cohomology of the $\mathbb{C}[X]$ in a (locally) closed $G$-stable subvariety give rise to equivariant $\mathcal{D}_X$-modules. Moreover, natural functors such as pushforwards and pullbacks via equivariant maps preserve equivariance of $\mathcal{D}_X$-modules. On the other hand, equivariant $\mathcal{D}$-modules are quite rigid when the group $G$ is large. When $G$ acts on $X$ with finitely many orbits, any coherent equivariant $\mathcal{D}_X$-module is automatically holonomic (see [HTT08, Theorem 11.6.1]). Moreover, in this case there are only finitely many simple equivariant holonomic $\mathcal{D}$-modules. It seems a reasonable task to classify and describe explicitly all equivariant $\mathcal{D}$-modules in such cases, which in general is considered to be a difficult problem [MV86, Open Problem 3].

1.2 Categories of equivariant $\mathcal{D}$-modules

Let $\text{mod}_G(\mathcal{D}_X)$ denote the category of equivariant coherent $\mathcal{D}_X$-modules. In [LW19a], we give a systematic study of $\text{mod}_G(\mathcal{D}_X)$ (e.g. [LW19a, Propositions 1.2, 1.4]). When $G$ acts on $X$ with finitely many orbits, the category $\text{mod}_G(\mathcal{D}_X)$ is equivalent to the category of finite-dimensional representations of a quiver with relations. This follows by the Riemann-Hilbert correspondence from the analogous result proved for perverse sheaves in [MV86, Vil94]. We give a more elementary and constructive proof of this result for $\mathcal{D}$-modules [LW19a, Theorem 3.4]. This constructive approach, together with various techniques developed in [LW19a], allow us to tackle the problem of finding the explicit description of the quiver (with relations) in many cases.
One such case is when $X$ is a variety with the action of a linearly reductive group $G$, and $B$ acts on $X$ with a dense orbit, where $B$ is a Borel subgroup of $G$. In [LW19a], we call such a space $X$ a $G$-spherical variety. For smooth spherical affine varieties, we show that several beautiful phenomena occur: any equivariant simple $\mathcal{D}$-module is multiplicity-free as a $G$-module, and its characteristic cycle is also multiplicity-free [LW19a, Theorem 3.16, Corollary 3.18]. For some spaces of matrices, explicit character formulas can be found in [Rai16]. Character formulas for other cases are being computed [Lörä]. In the case when $X$ is an irreducible $G$-module, we find explicitly the quivers corresponding to $\text{mod}_G(\mathcal{D}_X)$ [LW19a Theorem 5.2]. In almost all cases, the quiver of $\text{mod}_G(\mathcal{D}_X)$ is a disjoint union of quivers of type

\[
\widetilde{\mathbb{A}}_n : \ (1) \longrightarrow (2) \longrightarrow \ldots \longrightarrow (n-1) \longrightarrow (n) ,
\]

where all the 2-cycles are relations. In fact, the quiver $\widetilde{\mathbb{A}}_n$ has (up to isomorphism) only finitely many indecomposable representations, and they can be described explicitly [LW19a Theorem 2.11]. Only for the representation corresponding to $\text{Sp}_4 \otimes \text{GL}_4$ the quiver is the “doubling” of a different Dynkin quiver (see [LW19a Theorem 5.2 (b)]):

\[
\widetilde{\mathbb{E}}_6 : \begin{array}{c}
  \alpha \\
  \beta \\
\end{array} \\
\begin{array}{c}
  (1) \\
  (2) \\
  (3) \\
  (4) \\
  (5) \\
\end{array}
\]

where all the compositions with $\alpha$ or $\beta$ are relations, as well as all 2-cycles. Though not of finite representation type as $\widetilde{\mathbb{A}}_n$, the quiver $\widetilde{\mathbb{E}}_6$ is tame [LW19a Theorem 2.14].

Toric varieties are perhaps the most fundamental examples of spherical varieties. Studying equivariant $\mathcal{D}$-modules in this case is particularly important as it provides understanding of the $\mathcal{D}$-module structure of GKZ-systems. In [LW] we describe the quiver of the category for 3-dimensional normal toric varieties explicitly, and give a recursive construction in higher dimensional cases. We use results from the paper [RW18] that gives a formula for the weight filtrations provided by the mixed Hodge module structure of GKZ-systems.

We investigate similar problems beyond spherical varieties as well. On the space of binary cubic forms $\text{Sym}_3 \mathbb{C}^2$, in [LRW19] we described all the simple equivariant $\mathcal{D}$-modules and obtained the quiver of $\text{mod}_G(\mathcal{D}_X)$, whose largest connected component is the quiver

\[
\begin{array}{c}
  (1) \\
  (2) \\
  (3) \\
  (4) \\
  (5) \\
\end{array}
\]

where all 2-cycles and all non-diagonal compositions of two arrows are relations. This quiver is again of tame representation type [LRW19] Theorem 4.4]. Similarly, we describe all the simple equivariant $\mathcal{D}$-modules on the space of alternating senary 3-tensors
and on the other representations of the so-called subexceptional series \cite{LW19c}, where the quivers are a disjoint union of two type \(\tilde{A}_3\) quivers as in \eqref{eq:quiver}. This subexceptional series of representations comes from the third line of Freudenthal’s magic square associated to 5-dimensional symplectic geometries, see \cite{LM04}.

Another important source of examples comes from linear free divisors. Using our categorical methods, in \cite{LS} we are currently developing the explicit \(D\)-module and mixed Hodge module structures of the structure sheaf localized at a linear free divisor. A large class of linear free divisors is provided by quivers \cite{BM06}.

An indispensable ingredient in these results is the Bernstein-Sato polynomial (see \cite[Proposition 4.9]{LW19a}). We present details about these polynomials in Section 3.

Another idea of using \(D\)-modules is for the study a distinguished class of functions, called holonomic functions (for a survey, see \cite{Kau}). Many special functions are of this type, for example hypergeometric series or even the roots of polynomials (as functions in the coefficients, see \cite{Stu00}). Using equivariant \(D\)-modules, in \cite{Lorb} we describe a natural class of holonomic functions in an equivariant setting, coming from prehomogeneous vector spaces (e.g. binary cubic forms). In particular, we define the Bernstein-Sato polynomials of holonomic functions, and compute them for irreducible prehomogeneous spaces. This yields explicit tools to describe (meromorphic) flat connections on prehomogeneous vector spaces.

1.3 Applications to local cohomology

Exploiting our results on equivariant \(D\)-modules, we determine some local cohomology modules with support in orbit closures, expressing them as direct sums of indecomposable \(D\)-modules. Computing local cohomology modules is generally considered a difficult problem in commutative algebra and algebraic geometry. In the case of finitely many orbits we demonstrate the power of equivariant \(D\)-modules by obtaining results that are completely explicit.

In \cite{LR18}, we compute the (iterated) local cohomology modules with support in determinantal varieties. Some results were known in this direction (e.g. \cite{RW14}), but these did not provide a complete picture on the structure of local cohomology modules. Let \(X\) be the space of \(m \times n\) matrices, with \(m \geq n\). Let \(O_p\) denote the variety of matrices of rank \(p\), which is an orbit under the action \(G = \text{GL}_m \times \text{GL}_n\). Then \(X = \bigcup_{p=0}^{n} O_p\) has finitely many orbits under the action of \(G\). There are exactly \(n + 1\) simple equivariant coherent \(D_X\)-modules \(D_0, \ldots, D_n\), where the support of \(D_p\) is \(\overline{O}_p\). Note that \(D_n = \mathbb{C}[X]\) is the coordinate ring.

For \(a \geq b \geq 0\) we define the Gaussian binomial coefficient
\[
\binom{a}{b}_q = \frac{(1 - q^a) \cdot (1 - q^{a-1}) \cdots (1 - q^{a-b+1})}{(1 - q^b) \cdot (1 - q^{b-1}) \cdots (1 - q)}.
\]
Setting \(q \mapsto q^2\), this gives the Poincaré polynomial of Grass\((b, a)\). The following encodes the composition series of local cohomology of \(D_p\) in terms of generating functions.

\[
\binom{a}{b}_q = \frac{(1 - q^a) \cdot (1 - q^{a-1}) \cdots (1 - q^{a-b+1})}{(1 - q^b) \cdot (1 - q^{b-1}) \cdots (1 - q)}.
\]
Theorem 1.1 ([LR18 Theorem 1.1]). For every $0 \leq t < p \leq n \leq m$ we have the following in the Grothendieck group of $\text{mod}_G(\mathcal{D}_X)$:

$$\sum_{j \geq 0} [H^j_{G^i}(D_p)] \cdot q^j = \sum_{s=0}^t [D_s] \cdot q^{(p-t)^2 + (p-s)(m-n)} \cdot \binom{n-s}{p-s} \cdot \binom{p-1-s}{t-s} q^2.$$

As it turns out, in the non-square case $m \neq n$, the category $\text{mod}_G(\mathcal{D}_X)$ is semi-simple [ LW19a Theorem 5.4 (b)]. Hence, the above formula is actually a direct sum decomposition, and it gives all iterated local cohomology modules $H^j_{G^i}(\bigoplus_{j=0}^n (H^{k_1}_{G^{i_1}}(D_p) \cdots)).$

In the square case $m = n$, the situation is more complicated, as the category $\text{mod}_G(\mathcal{D}_X)$ is given by the quiver $\tilde{\text{A}}_{n+1}$ as in [1]. Let $S_{\text{det}}$ denote the localization of $S = \mathbb{C}[X]$ at the determinant. The Bernstein-Sato polynomial of $\det$ (see [3]) provides a filtration of $S_{\text{det}}$, from which we obtain the following indecomposable equivariant $\mathcal{D}$-modules $Q_p$ (with the corresponding representation of $\tilde{\text{A}}_{n+1}$)

$$Q_p := \frac{S_{\text{det}}}{(\text{det}^{p-n+1})}, \quad \mathbb{C} \cup \frac{1}{0} \cup \frac{1}{0} \cdots \cup \frac{0}{0} \cup 0 \cdots 0 \quad (p \text{ 1's})$$

Let $\text{add}(Q)$ denote the subcategory of $\text{mod}_G(\mathcal{D}_X)$ formed of $\mathcal{D}$-modules that are direct sums of $Q_0, Q_1, \ldots, Q_{n-1}$. In [LR18 Section 6] we show that for $t < p \leq n$, we have $H^j_{G^i}(D_p) \in \text{add}(Q)$, and we also have $H^2_{G^i}(Q_p) \in \text{add}(Q)$ $(p \neq n)$. Moreover, we obtain explicit $\mathcal{D}$-module direct sum decompositions for these local cohomology modules [LR18, Section 6]. Hence, we determine the direct sum decompositions of the iterations $H^j_{G^i}(\bigoplus_{j=0}^n (H^{k_1}_{G^{i_1}}(D_p) \cdots))$ for the square case as well.

In particular, we obtain all the Lyubeznik numbers $\lambda_{i,j}(R^{(p)})$ for the determinantal rings $R^{(p)}$, thus answering a question of Melvin Hochster (for a survey on Lyubeznik numbers, see [NBWZ16]). For the square case $m = n$, we have the following formula in terms of a bivariate generating function.

Theorem 1.2 ([LR18 Theorem 1.5]). We have $\sum \lambda_{i,j}(R^{(n-1)}) \cdot q^i \cdot w^j = (q \cdot w)^{n-1}$ and for $0 \leq p \leq n-2$ we have

$$\sum_{i,j \geq 0} \lambda_{i,j}(R^{(p)}) \cdot q^i \cdot w^j = \sum_{s=0}^p q^{s^2+2s} \cdot \binom{n-1}{s} q^2 \cdot w^{s^2+2p+s(2n-2p-2)} \cdot \binom{n-2-s}{p-s}.$$
In another direction, computing de Rham cohomology of local cohomology modules yields again interesting local invariants of varieties [Swi17]. Based partially on our previous results, we compute these for determinantal and Pfaffian varieties [LR]. The formulas obtained for determinantal varieties are similar to the formulas for the Lyubeznik numbers, but there is one substantial difference. Namely, the de Rham cohomology formulas are uniform with respect to both the non-square and square cases, and they encode information about the local cohomology modules that is dual in some sense to the information encoded by the Lyubeznik numbers.

**Theorem 1.3.** For \( m \geq n \), the de Rham cohomology groups of local cohomology modules are encoded by:

\[
\sum_{i,j \geq 0} h^i_{dR}(R^j_{Op}(S)) \cdot q^i \cdot w^j = \sum_{s=0}^{p} q^{(m-s)\cdot(n-s)} \cdot \binom{n}{s} \cdot w^{(n-p)\cdot(m-n)} \cdot \binom{n-1-s}{p-s} \cdot w^2.
\]

2 Geometric study of the representation theory of quivers

### 2.1 Introduction

Let \( Q \) be a quiver. For a fixed dimension vector \( \alpha \), the set of representations with dimension vector \( \alpha \) form an affine space \( \text{Rep}(Q, \alpha) \) that is a direct product of spaces of matrices. We investigate this space (and subvarieties) under the action of the product of general linear groups corresponding to the change of basis at each vertex. Under this action, there is a one-to-one correspondence between the isomorphism classes of representations \( V \) and orbits \( O_V \). The notion can be naturally extended to quivers with relations.

For an example, when \( Q \) is of type \( A_2 \) with dimension vector \((m, n)\), orbits correspond to ranks, and the orbit closures are precisely the determinantal varieties of \( m \times n \) matrices.

### 2.2 Orbit closures of quivers

The geometry of orbit closures \( \overline{O}_V \) has been studied intensively in various articles (for example [BZ01, BZ02, KR15, LM98]). In particular, it was shown that for Dynkin quivers of type \( A \) and \( D \) orbit closures have rational singularities, by reducing to the analogous facts on Schubert varieties. However, the type \( E \) case is still open. Furthermore, set-theoretic equations for orbit closures are known for all Dynkin quivers by [Bon96] and come from certain rank conditions. Nevertheless, finding the (minimal) defining equations and free resolutions of the orbit closures of Dynkin quivers is difficult in general.

We study these problems employing the geometric technique [Wey03] that is based on Kempf’s and Weyman’s generalization of Lascoux’s free resolution for determinantal ideals. In the case of Dynkin quivers, Reineke [Rei03] provides desingularizations of orbit closures \( \overline{O}_V \subset \text{Rep}(Q, \alpha) \) as total spaces of some vector bundles over a product of flag varieties which we will denote by Flag. To such a desingularization we associate a
locally free sheaf $\xi$, and apply in principle [Wey03, Basic theorem] to get a complex $F^\bullet$:

$$F_i = \bigoplus_{j \geq 0} H^j(\text{Flag}, \bigwedge^{i+j} \xi) \otimes A(-i-j).$$

(2)

Here $A$ denotes the coordinate ring of the representation space $\text{Rep}(Q, \alpha)$. When the complex $F^\bullet$ has no terms in negative degrees, the complex gives a minimal free resolution of the (normalization of the) coordinate ring of $\mathcal{O}_V$ over $A$.

The case when calculations in the complex $F^\bullet$ are possible using Bott’s theorem is when the locally free sheaf $\xi$ is semi-simple. This happens for 1-step representations (see [Sut15], or a more general definition in [LW19b]), when $\text{Flag}$ is just a product of Grassmannians. Generalizing the results in [Sut15], we give in [LW19b, Theorem 3.5] a beautiful connection between the geometry of 1-step orbit closures and the representation type of the quiver, on which the following result is based on.

**Theorem 2.1** ([LW19b, Theorem 3.6]). Let $Q$ be a Dynkin quiver and $V$ a 1-step representation. Then $\mathcal{O}_V$ has rational singularities (and is thus also normal and Cohen-Macaulay) and $F^\bullet$ gives a minimal resolution of its coordinate ring.

We have a similar result for extended Dynkin quivers [LW19b, Theorem 3.7], when normality fails in general, but the normalization still has rational singularities. This suggests a geometric trichotomy parallel to the representation-theoretic one—quivers of finite, tame and wild type.

In [LW19b, Section 4] we describe explicitly the minimal generators for the defining ideals of 1-step representations when the quiver is Dynkin of type $A$. Further, we show ([LW19b, Theorem 4.7]) that any representation can be written as a scheme-theoretic intersection of 1-step orbit closures. This provides an algorithm for finding an efficient generating set for the defining ideal of any orbit closure of a type $A$ quiver.

In [Lör20], we consider similar questions for the equioriented $A_3$ quiver. In this classical case, orbit closures correspond to varieties of pairs of matrices $(A, B)$ such that $\text{rank}(A) \leq a$, $\text{rank}(B) \leq b$ and $\text{rank}(A \cdot B) \leq c$, for some non-negative integers $a, b, c$. Here not all representations are 1-step. Hence, we cannot choose in general $\xi$ to be semi-simple in (2), and Bott’s theorem is not applicable directly. In [Lör20, Proposition 2.1] we turn the problem of finding the minimal free resolutions of orbit closures to the equivalent problem of calculating the cohomology groups of vector bundles on the 2-step flag $\text{Flag}(r_1, r_2, n)$ of the form

$$S_\lambda \mathcal{R}_2 \otimes S_\mu(W/\mathcal{R}_1)^*.$$

Here $S_\lambda$ (resp. $S_\mu$) denotes the Schur functor corresponding to the partition $\lambda$ (resp. $\mu$), $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) is the tautological subbundle of dimension $r_1$ (resp $r_2$), and $\dim W = n$. Computing the cohomology of such bundles is of independent interest, and can be seen as a first step towards the problem of computing cohomology of bundles on flag varieties that are not semisimple. In [Lör20, Section 3] we provide several methods to compute these cohomology groups, including a definite algorithm using Schur complexes. We
then apply the methods to obtain results about the geometry of the orbit closures of the equioriented A\textsubscript{3} quiver \cite[Section 4]{Lor20}.

2.3 Representation varieties of algebras with nodes

The operation of node splitting for finite-dimensional algebras was introduced in \cite{MV80}. Let Q be a quiver and consider its path algebra \( \mathbb{C}Q \). A node of an algebra \( A = \mathbb{C}Q/I \) is a vertex \( x \) of \( Q \) such that all the paths of length 2 passing strictly through \( x \) belong to the ideal \( I \). A node \( x \) of \( A \) can be split by the following local operation around \( x \) obtaining naturally an algebra \( A^x = \mathbb{C}Q^x/I^x \):

The representation theory of \( A \) and \( A^x \) are very closely related. In \cite{KL18}, we present various results on the relations between the geometry of varieties of representations in \( A \) and \( A^x \). In fact, we show that there is a natural map between varieties of representations from \( A^x \) to \( A \) that preserves the property of normality and rational singularities \cite[Theorem 1.2]{KL18}. The strength of our method lies in working in the “relative setting”, i.e. splitting locally a node without assuming any restrictions on the rest of the algebra. This is a very useful result for studying geometry of varieties of representations, as nodes are quite common in the representation theory of quivers (for example, each quiver in Section 1.2 has several nodes).

Results are particularly strong in the case of radical square zero algebras, which are algebras for which every vertex is a node. Splitting its nodes repeatedly we can describe all of its irreducible components in a purely combinatorial way \cite[Theorem 4.3]{KL18} and show that they have mild singularities.

**Theorem 2.2.** \cite[Corollary 1.4]{KL18} Let \( A \) be a finite-dimensional \( \mathbb{C} \)-algebra with \( \text{rad}^2A = 0 \). Then for any dimension vector \( \alpha \), any irreducible component \( C \subseteq \text{Rep}(A, \alpha) \) has rational singularities (and is thus also normal, and Cohen-Macaulay).

By results in \cite{CK18}, normality of the irreducibly components of \( \text{Rep}(A, \alpha) \) has strong consequences on the decomposition of its moduli spaces of its semistable representations in relation with Geometric Invariant Theory \cite[Section 5]{KL18}.

Our results are applicable to a wide class of algebras that we illustrate by numerous examples \cite[Examples 4.1, 4.6, 4.9, 4.11, 4.12, 5.4]{KL18}.
3 Bernstein-Sato polynomials

3.1 Introduction

Bernstein-Sato polynomials were introduced by J. Bernstein and M. Sato independently in the early seventies and have applications to singularity theory, monodromy theory etc. (for a survey, see [Bud]). Let \(f \in \mathbb{C}[x_1, \ldots, x_d]\) be a non-zero polynomial. The definition is based on the following existence result:

**Definition 1.** There is a differential operator \(P(s) \in D_X := D_X \otimes \mathbb{C}[s]\) and a non-zero polynomial \(b(s) \in \mathbb{C}[s]\) such that

\[
P(s) \cdot f^{s+1}(x) = b(s) \cdot f^s(x).
\]

Among such polynomials we call the (monic) polynomial \(b(s)\) of smallest degree the Bernstein-Sato polynomial (or \(b\)-function) of \(f\).

In general, the computation of \(b\)-functions of arbitrary polynomials is a difficult task. In the equivariant setting, several techniques have been developed. One case is that of prehomogeneous vector spaces (see [ST77]), that is, spaces that have a dense open orbit under the action of an algebraic group (indeed, this case was the original motivation for M. Sato). A semi-invariant is a polynomial invariant under the action of the group up to a character. An example in this setting is Cayley’s classical identity that gives the \(b\)-function of the determinant:

\[
det(\partial) \det(X)^{s+1} = (s+1)(s+2) \cdots (s+n) \det(X)^s.
\]

A lot of effort has been made to calculate \(b\)-functions of semi-invariants of prehomogeneous spaces (for example, see [Kim82, SKKO80]). In case of quivers, we call a dimension vector prehomogeneous if the corresponding representation space is prehomogeneous. For Dynkin quivers, each dimension vector is prehomogeneous. Semi-invariants of quivers have been studied extensively [DW00, Sch91, SdB01], and they are (sums of) determinants of matrices formed of suitable block matrices of variables.

3.2 \(b\)-functions of semi-invariants of quivers

In [Lör17, Lör19] we give two different techniques for the computation of the \(b\)-functions (of several variables) of various semi-invariants. For the case of quivers, both techniques generalize results obtained for type A quivers in [Sug11].

Our first technique is a slice method ([Lör19, Theorem 2.15]) that realizes a decomposition of the Bernstein-Sato polynomial of a semi-invariant into the product of two Bernstein-Sato polynomials - on associated to an ideal, the other to a smaller semi-invariant. This is based on our generalization ([Lör19, Theorem 2.5]) of a representation-theoretic multiplicity one property that was previously studied in [SS06]. In fact, this can be applied in other contexts and gives an elementary approach for the determination of the \(b\)-function of the determinant, symmetric determinant, Pfaffian and other
classical invariants (see [Lör19, Section 2.3]). The technique is efficient for computing the \( b \)-functions of many semi-invariants, including for tree quivers of “small weights” [Lör19, Theorem 3.13, Theorem 3.14].

The other technique is based on relating \( b \)-functions under the so-called reflection functors (also known as Coxeter or castling transformations) fundamental to representation theory. For quivers, this is an operation performed at a sink (resp. source) vertex by changing the orientation of all the arrows ending (resp. starting) at the vertex. We generalize a formula in [Kim82] and obtain a relation between the \( b \)-functions of semi-invariants related under reflection functors [Lör17, Theorem 2.1, 2.2, 4.1]. Namely, if we put
\[
[s]_{α_1}^{d_1}...d_k := \prod_{i=1}^{a} \prod_{j=0}^{d_i-1} (d_1 s_1 + \cdots + d_k s_k + i + j),
\]
then the following holds.

**Theorem 3.1 ([Lör17 Theorem 4.1]).** Let \( Q \) be quiver with a prehomogeneous dimension vector \( β \) and \( f_i \in \text{SI}(Q, β)_{(α_i, \cdot)} \) be semi-invariants, where \( i = 1, \ldots, k \). Assume the coordinates of \( c(β) \) are non-negative. Then the \( b \)-function satisfies the formula
\[
b_f(s) = b_{c(f)}(s) \prod_{x \in Q_0} [s]_{c(α_1)_x}^{c(α_1)_x}...d_k s_k^{c(α_k)_x}.
\]

This allows, in particular, the computation of \( b \)-functions (of several variables) for all Dynkin quivers [Lör17 Corollary 4.3] as well as extended Dynkin quivers with prehomogeneous dimension vectors [Lör17 Proposition 4.5]. For examples, see [Lör17 Example 4.6, 4.7, 4.8, 4.9].

### 3.3 Singularities of zero sets of semi-invariants for quivers

Bernstein-Sato polynomials provide fine numerical invariants of singularities (for example, cf. [BMS06, Sai93]). Using our Theorem 3.1 and representation theory of quivers, we have show that codimension 1 orbit closures in for Dynkin quivers have rational singularities [Lör15, Theorem 3.5] and give a similar result for extended Dynkin quivers [Lör15, Theorem 3.7].

In [BMS06] the notion of the Bernstein-Sato polynomial has been generalized to arbitrary varieties. In [Lör15 Proposition 3.2] we give a basic relation between the \( b \)-function of several variables of some semi-invariants and the Bernstein-Sato polynomial of their zero set. Using this, we give some results when nullcones have rational singularities [Lör15 Proposition 3.10, Example 3.12]. In order to prove these claims using Bernstein-Sato polynomials, one has to work with reduced complete intersections [BMS06 Theorem 4]. It was shown in [RZ03] that nullcones indeed become (irreducible) complete intersections when the prehomogeneous dimension vector is not “too small”. We prove that they also become reduced:
Theorem 3.2 ([Lör15, Theorem 2.7]). Let $Q$ be a quiver and $\alpha$ a prehomogeneous dimension vector. Then there is a positive integer $N$ such that the nullcone in $\text{Rep}(Q, n \cdot \alpha)$ is reduced for any $n \geq N$.

We give a bound for $N$ in [Lör15, Theorem 2.10] for Dynkin quivers that is sharp, as we found a type $E_8$ quiver where the nullcone is not reduced [Lör15, Example 2.12].

3.4 Bernstein-Sato polynomial for maximal minors

Conjectures have been made on the Bernstein-Sato polynomials of minors of a generic $m \times n$ matrix of variables (see [Bud]). These have been disproved in general, but we confirmed them in the case of maximal minors [LRWW17]. Let $m \geq n$. We show

Theorem 3.3 ([LRWW17, Theorem 4.1]). The Bernstein-Sato polynomial of the ideal of maximal minors equals $(s + m - n + 1) \cdots (s + m)$.

There are two main steps in the proof of this result. The first part is finding the equation giving the predicted Bernstein-Sato polynomial. We have done this in the more general setting of so-called multiplicity-free tuples ([LRWW17, Proposition 3.4]). From this we computed the predicted polynomial (see [LRWW17, Theorem 3.5]). In [LRWW17, Section 4] we prove that the predicted polynomial indeed is minimal, therefore it is the Bernstein-Sato polynomial.

The method works also for the case of sub-maximal Pfaffians ($n$ odd):

Theorem 3.4 ([LRWW17, Theorem 3.9]). The Bernstein-Sato polynomial of the ideal of sub-maximal Pfaffians equals $(s + 3)(s + 5) \cdots (s + n)$.

We also give connections between roots of Bernstein-Sato polynomials and the non-vanishing of certain local cohomology groups [LRWW17, Corollary 3.2].

Finally, we conclude that the Strong Monodromy Conjecture holds for maximal minors and sub-maximal Pfaffians [LRWW17, Section 5].

References


[Löra] ______: Characters of equivariant D-modules on affine spherical varieties, in progress.


