# Stochastic Resonance in a System of Coupled Asymmetric Resonators

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**Abstract.** A system of three two-state resonators subject to noise and a harmonic signal is investigated. In both cases of either symmetric or asymmetric resonators a non monotonous dependence of the spectral output on the coupling strength is found, although for different kinds of coupling. Whereas the amplification of the signal for symmetric coupled resonators is optimized for positive coupling (attractive interaction), in the strong asymmetric case the optimal output is found for negative coupling (repelling interaction) due to a symmetrization of the system by the coupling.

## 1 Introduction.

The detection of a weak signal by a nonlinear resonator can be improved by an appropriate level of noise. This is the essential point of *stochastic resonance* (SR) [for extensive reviews, see [1] and [2]].Consider a bistable resonator driven by noise and a weak harmonic signal, for instance the overdamped Brownian motion in a double well potential. In that case, the power spectral density of the system consists of a typical noisy background and peaks at the driving frequency and its higher harmonics. The height of these peaks scaled with the signal intensity is called the spectral power amplification (SPA) and passes through a maximum as a function of noise intensity. The ratio of the peak height to the noisy background, the signal-to-noise ratio (SNR) exhibits the same feature which is a manifestation of the SR effect.

In recent years it was shown that SR can be significantly enlarged in an array of coupled stochastic resonators instead of just a single element. For this *array enhanced stochastic resonance* the coupling strength turned out to be a second design parameter with regard to the purpose of signal detection. This has been demonstrated with numerical simulations [3] as well as analytical studies [4] [5].

The analytical problem of calculating the SPA and SNR, became tractable in [4] [5] by two simplifications: 1) a reduction of the (generally continuous) bistable dynamics of the single resonator to a two-state model and 2) the assumption of a spin like nearest neighbor interaction between these two-state resonators. The first approximation was originally worked out in an early theory of SR [6], while the second assumption permitted the use of Glauber's theory for the stochastic Ising model of a spin chain. The analytical results for an infinite number of resonators coupled in a chain revealed indeed the aforementioned twofold resonance with respect to the noise and coupling strength. As an example for finite system, three coupled symmetric resonators were studied in [7] exhibiting the same effect.

Of course, systems displaying SR are not necessarily symmetric. It might thus be justified to ask for the influence of an asymmetry in the transition rates of a chain of coupled two-state resonators. In this lecture, we will study such a system. For simplicity, we restrict our consideration to the case of three identical elements driven by the same harmonic signal. We will focus on the spectral power amplification as a measure for stochastic resonance.

In the following section, we shall motivate the transition rates of the model in detail and explain how to obtain the SPA of a single coupled resonator from the master equation of the system. The analytical result for the spectral power amplification will be compared to numerical simulations. Furthermore, differences to the case of symmetric elements will be discussed.

## 2 Model

The system we are going to investigate is illustrated in Fig.1. Each of the three resonators can be in one of two states $\sigma = \pm 1$ corresponding to the minima of the double well potential. Apart from the interaction, the transition rates are the same for all resonators but clearly asymmetric since the right well is deeper than the left one, i.e. the transition rate to the right will be generally larger than that to the left. In the absence of coupling and signal these rates are given by a Kramers law [8]

$$\begin{aligned} W_i^0 &\sim \frac{\alpha}{2} = \frac{\alpha_0}{2} \exp(-(\Delta U - B)/D) \\ W_i^0 &\sim \frac{\beta}{2} = \frac{\beta_0}{2} \exp(-(\Delta U + B)/D) \end{aligned} \tag{1}$$

for transitions from left to right and vice versa, respectively. $\Delta U \pm B$ denotes the respective potential barrier with $B$ as the asymmetry parameter, $D$ stands for the noise strength and the constants $\alpha_0, \beta_0$ are set to unity for the sake of simplicity. Throughout this work we fix the potential barrier to $\Delta U = 0.25$. The coupling we adopt from [4] [5] [9] as a nearest neighbor interaction, for a system of three elements this means that every resonator is coupled to every other one. The transition probabilities can generally be written as

$$W_i^0(\sigma_i) = \left(\frac{1}{4}(\alpha + \beta) + \frac{1}{4}\sigma_i(\beta - \alpha)\right)\left(1 - \frac{\gamma}{2}\sigma_i(\sigma_{i+1} + \sigma_{i-1})\right) \tag{2}$$

A positive coupling strength $\gamma$ implies an attractive interaction, the resonators are forced to align in the same well. In analogy to a spin system this case is often referred to as ferromagnetic coupling. For the particular

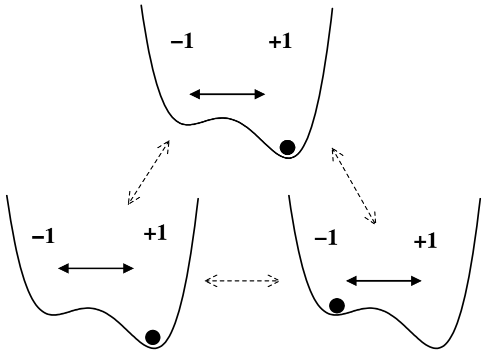


**Fig. 1.** Three coupled resonators (filled circles) are subject to identical asymmetric double well potentials, to white noise and to an external signal. The states $-1$ and $+1$ correspond to the left and right well, respectively.

configuration depicted in Fig. 1, this would mean that the lower right resonator being in state $\sigma = -1$ experiences an attraction towards the other state. In contrast, no force is obviously exerted on the other resonators because the interactions of neighboring elements compensate each other. For a negative (anti ferromagnetic) coupling ($\gamma < 0$) the resonators are driven into an anti parallel alignment. In that case, the lower right element in Fig. 1 has a tendency to remain in $\sigma = -1$ longer than in the absence of coupling.

Assuming detailed balance for the rates (2) the parameter $\gamma$ is connected to the noise strength and to an independent parameter $J$ [4] [5] [9] henceforth referred to as coupling strength

$$\gamma = \tanh\left(\frac{2J}{D}\right). \tag{3}$$

Note, that the value of $\gamma$ is limited to the range $[-1, 1]$ and its sign coincides with that of $J$.

An additive weak and sufficiently slow signal modulates the potential barriers, suppressing or enhancing the transitions dependent on the state which is currently occupied. Thus, the Kramers rates [8] become a function of time [6]. An expansion in leading order of the signal amplitude $A$ then yields the time dependent transition rates

$$W_i(\sigma_i) = W_i^0(\sigma_i)\, e^{-\sigma_i \frac{A}{D}\cos(\Omega t+\phi)} \approx W_i^0(\sigma_i)\left(1 - \sigma_i \frac{A}{D}\cos(\Omega t + \phi)\right). \tag{4}$$

with $\Omega$ standing for the signal frequency and $\phi$ the initial phase of the signal.

The master equation of the system reads

$$\dot{P}(\boldsymbol{\sigma}) = \sum_{i=1}^{3} (F_i - 1)\, W_i(\sigma_i)\, P(\boldsymbol{\sigma}) \tag{5}$$

with the operators $F_i$ defined by

$$F_i f(\sigma_i) = f(-\sigma_i) \tag{6}$$

for $i = 1, 2, 3$ and the rates $W_i(\sigma_i)$ given by (4).

Although the single resonator is asymmetric this asymmetry is the same for all resonators. Since we are only interested in long time properties of the system we can utilize the *translational invariance* of it, i.e. if the initial conditions are forgotten all resonators obey the same dynamics.

In order to obtain the SPA $\eta$ it is necessary to calculate the time dependent mean value of the single resonator[10] [4] [5]. In principle it is possible to obtain this time dependent mean value by direct solution of the equation (5). This can be done by rewriting the master equation as a set of eight (linear and time dependent) differential equations corresponding to the possible states of the three resonators.

Another way which strictly follows Glauber's work is to obtain directly from (5) the equation for the time dependent equal time correlators. Using this method together with the assumption of the translational invariance the whole problem is to a large extent simplified. For the first three asymptotic equal time correlators we use in the following the abbreviations

$$s(t) = \langle \sigma_i(t) \rangle, \qquad r_1(t) = r_{k\pm 1}(t), \qquad c(t) = \langle \sigma_1(t)\sigma_2(t)\sigma_3(t) \rangle. \tag{7}$$

Here, we have utilized the translational invariance resulting in the independence of the first equal time correlator on the position in the ring and the dependence of the second on the distance between the resonators only.

## 3 Symmetric case

This case was already studied in [7]. Here, the asymmetry parameter $B$ in eq.(1) is equal to zero and the transition rates are given by

$$W_i(\sigma_i) = \frac{1}{2}\alpha\left(1 - \frac{\gamma}{2}\sigma_i(\sigma_{i+1} + \sigma_{i-1})\right)\left(1 - \sigma_i \frac{A}{D}\cos(\Omega t + \phi)\right). \tag{8}$$

By multiplication of eq.(5) with $\sigma_i$, $\sigma_i\sigma_k$, $\sigma_i\sigma_k\sigma_l$ and averaging one obtains the following correlator equations [4] [5] [9]

$$\begin{aligned}
\frac{ds}{dt} &= -\alpha(1-\gamma)s(t) + \alpha\frac{A}{D}(1-\gamma r_1)\cos(\Omega t + \phi) \\
\frac{dr_1}{dt} &= -\alpha(2-\gamma)r_1(t) + \alpha\gamma + \alpha\frac{A}{D}[(2-\gamma)s(t) - \gamma c(t)]\cos(\Omega t + \phi) \\
\frac{1}{3}\frac{dc}{dt} &= -\alpha c(t) + \alpha\gamma s(t) + \alpha\frac{A}{D}(1-\gamma)r_1(t)\cos(\Omega t + \phi).
\end{aligned} \tag{9}$$

These equations can be solved in linear response theory ($A \ll 1$) by making the following ansatz

$$s = s_0 + As_1, \qquad r_1 = r_{1_0} + Ar_{1_1}, \qquad c = c_0 + Ac_1. \tag{10}$$

where the expressions with subscript 0 denote the unperturbed solutions (A=0). The long time limit of the solution for the mean value of the single resonator reads

$$s(t) = q\cos\left(\Omega t + \phi + \theta\right) \tag{11}$$

where amplitude $q$ and phase shift $\theta$ are given by

$$q = \frac{A}{D}\alpha\left(1-\gamma\right)\frac{2+\gamma}{2-\gamma}\frac{1}{\sqrt{\Omega^2+\alpha^2\left(1-\gamma\right)^2}}, \qquad \tan\theta = \frac{-\Omega}{\alpha\left(1-\gamma\right)}. \tag{12}$$

After averaging over the uniformly distributed initial phase $\phi$, the SPA $\eta$ is obtained as

$$\eta_s = \frac{q^2}{A^2} = \frac{1}{D^2}\alpha^2\left(1-\gamma\right)^2\left(\frac{2+\gamma}{2-\gamma}\right)^2\frac{1}{\left(\alpha\left(1-\gamma\right)\right)^2+\Omega^2}. \tag{13}$$

In Fig. 2 the SPA is presented for different values of the (positive) coupling strength $J$ and two different frequencies. The signal amplification curve possesses maxima both with respect to the noise intensity and the coupling constant $J$. As already reported in [7], this non monotonous dependence on the coupling vanishes for negative $J$. From Fig. 2 it can be seen that the noise intensity maximizing SPA $D_{max}$ shifts with increasing coupling to larger values.

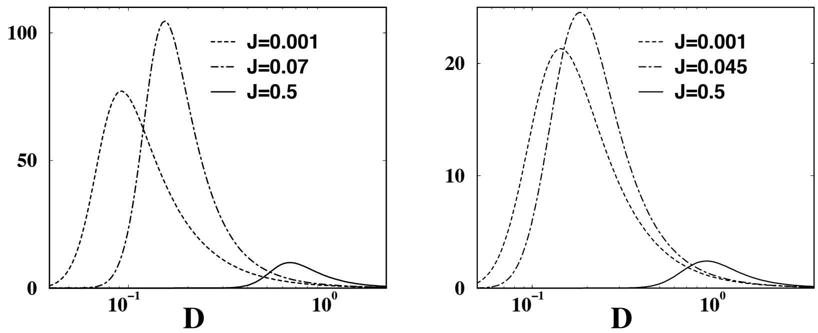


**Fig. 2.** SPA versus noise intensity in the system of the three coupled symmetric resonators for different coupling constant and values of $\Omega = 0.05$ (left panel) and $\Omega = 0.2$ (right panel).

In order to find the conditions for the global maximum of the SPA with respect to the coupling and noise strength, the partial derivatives of the SPA with respect to $J$ and $D$ can be calculated. This yields the following conditions

$$\Omega^2 = e^{\left(\frac{-2\Delta U}{D}\right)} \frac{\left(32\Delta U^2 + 28\Delta U D + 13D^2\right) - \gamma_{max}\left(8\Delta U^2 + 16D\Delta U + 7D^2\right)}{D^2 \frac{1}{2}\gamma_{max}\left(2 - \frac{1}{2}\gamma_{max}\right)} \tag{14}$$

and

$$\gamma_{max} = \frac{1}{D}\left(\Delta U - \sqrt{\Delta U^2 + D^2 - D\Delta U}\right). \tag{15}$$

By evaluation of eq. (14) and (15) it can be confirmed that an overall maximum with respect to $D$ and $J$ can occur for a positive coupling strength $J$ only as already mentioned above.

The static response of the system i.e. the amplification for $\Omega = 0$ reads

$$\eta_s^{static} = \frac{1}{D^2}\left(\frac{2+\gamma}{2-\gamma}\right). \tag{16}$$

For all values of the coupling constant $J$ the static response diverges with decreasing noise intensity. The SPA for a finite frequency can be recast into

$$\eta_s = \eta_s^{static}\frac{1}{1+d^2} \tag{17}$$

where $d = \frac{\Omega}{\alpha(1-\gamma)}$ is the dynamical factor which causes the decrease of the SPA for small noise intensities [4]. For large noise intensity since $d \ll 1$ the behavior of the SPA is determined by the static response. Contrary, for small noise intensity $d \gg 1$ i.e.

$$\eta_s \approx \eta_s^{static}\frac{1}{d^2}. \tag{18}$$

This yields the matching condition for the value of the noise which maximizes the SPA

$$\Omega \approx \alpha\left(D_{max}\right)\left(1 - \gamma\left(J, D_{max}\right)\right) \tag{19}$$

or

$$J \approx \frac{D_{max}}{2}\operatorname{arctanh}\left(1 - \Omega\exp\left(\frac{\Delta U}{D_{max}}\right)\right). \tag{20}$$

For a given frequency $\Omega$ the latter formula allows the conclusion that $J$ is a monotonously increasing function of $D_{max}$ and vice versa.

The physical mechanism of the enhancement of the amplification versus coupling constant can be understood as follows. For small coupling each resonator is able to follow the signal independently. With increasing coupling the particles start to synchronize i.e. when two of them follow the external signal they can by the attractive force make the third particle behave in the same way. For increasing coupling constant a transition of the single resonator synchronized with the signal becomes a rare event. The possibility of simultaneous transition of all resonators is not taken into account by the Markov approach of the master equation (5).

## 4 Asymmetric case

In the asymmetric case ($B \neq 0$) the differential equations for the equal time correlators read

$$2\frac{ds}{dt} = -(\alpha+\beta)(1-\gamma)s(t) + (\beta-\alpha)\Big(\gamma r_1(t) - 1\Big)$$
$$+\frac{A}{D}\Big[\,(\beta-\alpha)\,(1-\gamma)\,s(t) - (\alpha+\beta)\Big(\gamma r_1(t) - 1\Big)\Big]\cos(\Omega t+\phi]$$

$$2\frac{dr_1}{dt} = -(\alpha+\beta)\Big(\,(2-\gamma)r_1(t) - \gamma\Big) + (\beta-\alpha)\Big(\gamma c(t) - \,(2-\gamma)s(t)\Big)$$
$$+\frac{A}{D}\Big[\,(\alpha+\beta)\Big(\,(2-\gamma)s(t) - \gamma c(t)\Big) + (\beta-\alpha)\Big(\,(2-\gamma)r_1(t) - \gamma\Big)\Big]\cos(\Omega t+\phi)$$

$$\frac{2}{3}\frac{dc}{dt} = -(\alpha+\beta)\Big(\,c(t) - \gamma s(t)\Big) - (\beta-\alpha)(1-\gamma)r_1(t)\gamma s(t)$$
$$+\frac{A}{D}\Big[\,(\alpha+\beta)(1-\gamma)r_1(t) + (\beta-\alpha)\Big(\,c(t) - \gamma s(t)\Big)\Big]\cos(\Omega t+\phi). \quad (21)$$

Again, a linear ansatz like in (10) can be made. Here, the problem of determining the oscillatory part of $s(t)$ is much more complicated since after the substitution of (10) into (21) the resulting equations are still coupled.Therefore, the formula for the SPA becomes rather lengthy and is not presented here. The final evaluation has been performed by means of a symbol manipulating computer program.

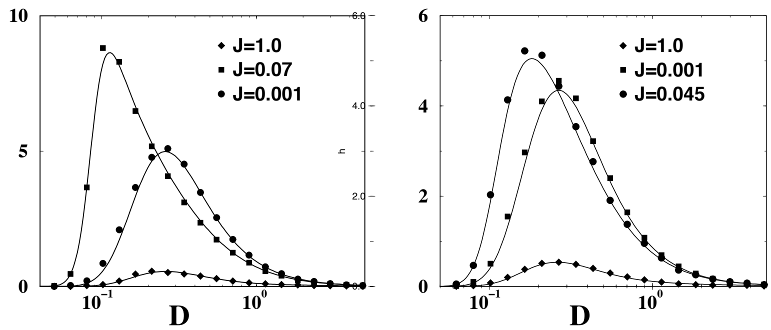


**Fig. 3.** SPA versus noise intensity for $B = 0.2$ and two different values of the input frequency $\Omega = 0.05$ (left panel) and $\Omega = 0.2$ (right panel). The analytical curves are compared to results of cumputer simulations of eq. (5) at signal amplitude $A = 0.05$.

Consider first the case of a strong asymmetry. In this case, no enhancement of the SPA with respect to the coupling is observed for *positive values* of $J$ contrary to the symmetric case. The effect occurs now for *negative* coupling as it is shown in Fig. 3 for two different driving frequencies. There exists a moderate coupling strength $J$ for which the SPA attains a maximum the height of which is quite small compared to the symmetric case (cf. the scales in Figs. 2 and 3, respectively). Note that the maximum with respect to both coupling and noise strength appears at a fairly small noise strength, while for larger or smaller values of $J$ the maximum with respect to $D$ is observed at larger values of $D$.

In order to verify these complicated features of the asymmetric system we have performed computer simulations of eq. (5). Using power spectra of a set of system trajectories ($10^{18}$ time steps, amplitude $A = 0.05$) we estimated the SPA also shown in Fig. 3. The analytical curves and the simulation data agree rather well and thus reveal the validity of the linear response ansatz and of our results.

Consider the SPA for changing asymmetry depicted in Fig. 4 as a function of the noise intensity and the coupling constant. For each value of $B$ there appears a global maximum which generally decreases and shifts to negative values of the coupling parameter $J$ with increasing asymmetry $B$. For $B \approx 0$ (symmetric case) the global maximum of the SPA occurs at a positive coupling strength $J_{max}$ as discussed above. For moderate values of $B \approx 0.1$ the maximum appears at $J_{max} = 0$, in this case an improvement of the SPA by the coupling of the three resonators is obviously not possible. For

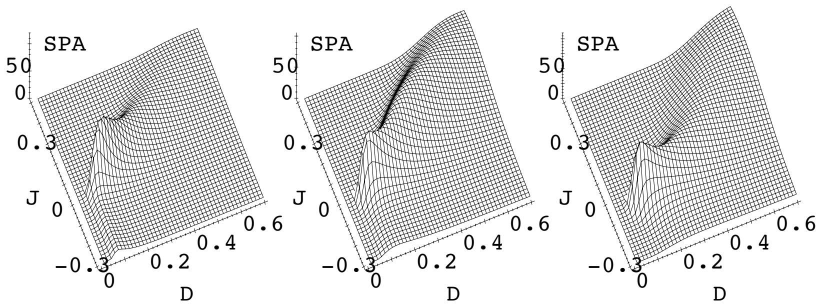


**Fig. 4.** SPA as a function of the coupling constant $J$ and noise intensity $D$ for fixed frequency $\Omega = 0.05$ for three different values of the asymmetry parameter $B = 0$,$B = 0.1$ and $B = 0.2$

large asymmetry $0.1 < B < \Delta U$ the maximum is observed at a negative coupling constant $J_{max} < 0$ as mentioned above. The mechanism of the enhancement differs essentially from that one in the symmetric case. In case of a large asymmetry a repelling coupling enables the system to operate in a *symmetric* way, i.e. the coupling *removes* effectively the asymmetry *and* the suppression of the response being characteristic for asymmetric systems [1]. We will explain this symmetrization now in detail.

For small coupling constant and small noise intensity all three resonators are in the deeper right well (see Fig. 1). This is the dominant state of the system. With growing repelling interaction, i.e. with increasing *negative* $J$ another configuration becomes possible: one resonator being in the left well and two in the deeper right well. The rate for the transition to this configuration is $\beta(1+|\gamma|)/2$, since there are three possibilities the total rate is multiplied by three. The rate for the inverse process is $\alpha(1-|\gamma|)/2$, for a certain value of the coupling these two rates can be equal, i.e. the system is at least with regard to this two transitions symmetric. This corresponds to

$$\alpha(1-|\gamma|)/2 = 3\beta(1+|\gamma|)/2 \tag{22}$$

from which we obtain

$$|J| = \frac{1}{2}\left(B - D\frac{\ln(3)}{2}\right). \tag{23}$$

Of course, the two configurations are not the only ones, but are rather dominant as can be seen from the trajectories depicted in Fig. 5. In the beginning

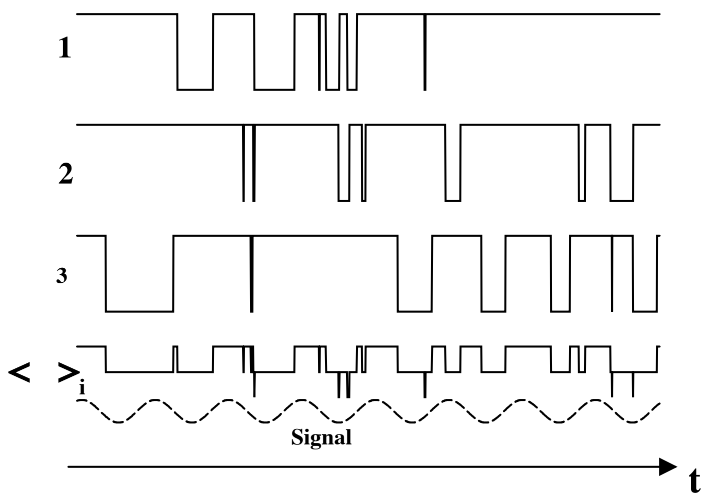


**Fig. 5.** Sample trajectory of the system for a parameter set according to the symmetry condition eq. (23). From Top to bottom: trajectories of $\sigma_1, \sigma_2, \sigma_3$, their average $\langle\sigma\rangle_i$ and the signal (dashed line). $\langle\sigma\rangle_i$ jumps between 1 and 1/3 and once in a while also to $-1/3$ correponding to a configuration with two resonators being in the left well. In order to illustrate the synchronization with respect to the signal a sufficiently large amplitude was used. Parameters: $J=-0.07, B=0.2, \Omega=0.05, A=0.09, D=0.1$.

all three resonators are in the right well ($\sigma_i = 1$), then the third resonator jumps to the left well ($\sigma_1 = -1$), goes back to the right one, the first is now jumping to the left and so on. The two configurations and the particular symmetry of the system can be readily seen by means of the “average” of the three resonators $\langle\sigma\rangle_i = (\sigma_1 + \sigma_2 + \sigma_3)/3$. Regardless of the specific resonator jumping to the left and back this quantity takes mostly the values $\langle\sigma\rangle_i = 1$ (all in the right well) and $\langle\sigma\rangle_i = 1/3$ (one in the left well) with almost equal probability. Once in a while there are also transitions to a configuration with *two* resonators in the left well ($\langle\sigma\rangle_i = -1/3$) from which the system quickly gets back to $\langle\sigma\rangle_i = 1/3$. Apart from these rare events the average $\langle\sigma\rangle_i$ is symmetric and can follow the likewise symmetric signal $\cos(\Omega t + \phi)$ in an optimal way. The average of the *single resonator* is due to the translational invariance of the system equal to a third of $\langle\sigma\rangle_i$, hence in this case the SPA is approximately equal to that one of a symmetric system with rates (22) multiplied by a prefactor 1/9. We want to point out that this holds only true for values of $D$ and $J$ obeying eq. (23).

Using the formula for the symmetric single resonator with modified rates (22) and taking account of the prefactor

$$SPA = \frac{1}{9}\frac{1}{D^2}\frac{[\alpha(1-|\gamma|)]^2}{[\alpha(1-|\gamma|)]^2 + \Omega^2} \tag{24}$$

we find for the maximal SPA the following condition

$$\Omega^2 = \frac{[\alpha(1-|\gamma|)(\mu+1)]^2}{\mu^2(\frac{\Delta U - B}{D} - 1) + 2\mu(\frac{\Delta U}{D} - 1) + (\frac{\Delta U + B}{D} - 1)} \tag{25}$$

with $\mu = (1-|\gamma|)/(1+|\gamma|)$.

Eqs. (23) and (25) are approximative implicit conditions for the global maximum. For the set of parameters corresponding to Fig. 3 we find $D = 0.10922$, $\Omega = 0.053$ where we have used in fact a driving frequency of $\Omega = 0.05$ and the global maximum appears at $D_{max} \approx 0.11$. The small deviation in the frequency is due to the fact that the above argumentation holds true for small noise and hence for small driving frequencies only. For larger noise also transitions to the other states (two resonators in the left well) occur more often and have to be taken into account. These transitions result in a slightly larger SPA than expected by the approximation (24). The reason for this deviation is that more than one resonator can follow the signal within one period.

## 5 Summary

We have investigated a periodically modulated system of three coupled resonators by means of the spectral output of a single resonator and have shown that either a system of symmetric as well as asymmetric resonators can exhibit an enhancement of the output by tuning the coupling strength. While in the symmetric system this non monotonous dependence of the SPA turned out to be due to a cooperative effect between the three resonators, in the asymmetric case a completely different mechanism applies. We could demonstrate that in this case the asymmetry of the system can be effectively removed by an appropriate repelling coupling. The analytical results were found to be in good agreement with numerical simulations of the full system.

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