

Noise-Induced Transport with Low Randomness

Benjamin Lindner

Department of Physics, University of Ottawa, 150 Louis Pasteur, Ottawa, Canada K1N 6N5

Lutz Schimansky-Geier

Institute of Physics, Humboldt-University at Berlin, Invalidenstrasse 110, D-10115 Berlin, Germany

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We study the transport of overdamped Brownian particles in periodic potentials subject to a spatially modulated Gaussian white noise. We derive an analytical expression for the diffusion coefficient of particles. By means of velocity, diffusion coefficient, and their ratio (Péclet number) we discuss (a) symmetric potential and modulation of noise intensity and (b) a ratchet profile with strong noise modulation. It is shown that state dependent fluctuations may not only induce directed transport, but also a pronounced coherence of transport if the potential exhibits a strong asymmetry.

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The rectification of fluctuations by means of stochastic ratchets manifests itself by a finite mean velocity or finite particle current, even if macroscopic gradients of the system entirely vanish. This noise-induced directed transport has been the subject of many theoretical and experimental studies addressing important issues as, for instance, efficiency of transport, the influence of inertia or internal degrees of freedom, motion in higher dimensional ratchet potentials, collective effects in large assemblies of coupled ratchets, and the possibility of current reversals upon variation of the system's parameters (see the recent review [1] and references therein).

Another important question that has received only little attention so far is that of coherence of transport. A noise-induced directed motion is necessarily accompanied by diffusive spreading among different realizations of the process. The latter may completely overshadow the ratcheting effect in a system with *finite* spatial extensions. Given that in many systems finite-size restrictions are more severe than, for instance, restrictions in energy supply, a high transport coherence (i.e., transport with low level fluctuations) may become crucial to observe a directed transport at all. Transport coherence can be quantified by diffusion coefficient and Péclet number (i.e., the ratio of velocity and diffusion coefficient) which are *nonstationary* characteristics of transport.

In the few ratchet models where diffusion coefficient or Péclet number were studied [2,3] randomness of transport was high; Péclet numbers in [2] were always well below 0.2, in [3], even superdiffusive behavior corresponding to zero Péclet number was observed. In contrast to these findings, recent experiments on molecular motors [4,5] report on low randomness, i.e., high transport coherence, corresponding to Péclet numbers between 2 to 6.

The aim of this Letter is to provide a simple theoretical example for noise-induced transport with low randomness. We will use a model for Brownian motion in a periodic potential with likewise periodically modulated noise intensity termed the Seebeck ratchet in Ref. [1].

First, we analyze the classic example by Büttiker [6] where the asymmetry required for directed motion originates from a phase shift between a *symmetric* potential and noise modulation. In this case the coherence of transport will be fairly low. Next, we will show that an additional asymmetry in the potential results in a high coherence of the motion indicated by a high Péclet number and a minimum of the diffusion coefficient at an optimal value of the noise intensity. All results are based on novel analytical results which are confirmed by means of numerical simulations.

Model.—We consider the overdamped dynamics of a Brownian particle with spatial coordinate x given by

$$\dot{x} = f(x) + g(x)\sqrt{2D}\xi(t), \quad (1)$$

where the potential $U(x)$ corresponding to the force $f(x)$ by $f(x) = -dU(x)/dx$ as well as the non-negative noise modulation $g(x)$ are periodic functions of the same period L_x , i.e.,

$$U(x) = U(x + L_x), \quad g(x) = g(x + L_x), \quad g(x) > 0. \quad (2)$$

The model Eq. (1) will always be interpreted in the sense of Stratonovich. Systems with multiplicative noise such as Eq. (1) are governed by an *effective potential* [7]

$$\Psi(x) = - \int_0^x d\tilde{x} \frac{f(\tilde{x})}{g^2(\tilde{x})}. \quad (3)$$

Noise-induced transport occurs if and only if this potential is *not* a periodic function, i.e., exhibits an effective bias [6,8]

$$\Psi(L_x) - \Psi(0) \neq 0 \Leftrightarrow v_x = \langle \dot{x} \rangle \neq 0. \quad (4)$$

Such a bias can be induced by asymmetries of potential and noise modulation [9] *or* by a suitable phase shift between *entirely symmetric* potential and noise modulation. The latter was demonstrated by Büttiker [6] and others [8]. The potential and noise modulation he considered were given by

$$U(x) = 1 - \cos(x), \quad g(x) = \frac{1}{\sqrt{1 - \alpha \cos(x - \phi)}}. \quad (5)$$

Throughout this work we use $\alpha = 0.95$ and $\phi = 1$ [cf. Fig. 1(a)] leading to a finite bias in the effective potential Eq. (3). The physical origin of this bias and the finite mean velocity is that a high noise intensity at one slope of the potential causes a higher escape probability compared to that at the opposite slope. In the presented situation the Brownian particle will go to the right on average. This is illustrated in Fig. 2(a) by means of five sample trajectories all started in $x(t=0) = 0$. Note the considerable spreading among the trajectories.

The second system with a similar phase shift between the potential and the noise modulation but with an additional spatial asymmetry in the potential is given by

$$U(x) = A(\beta) \sin(x) e^{\beta[\cos(x)-1]}, \quad g(x) = e^{\beta[\cos(x)-1]/2}. \quad (6)$$

The prefactor $A(\beta)$ is chosen such that the potential barrier equals 2 as for the Büttiker ratchet. For small β the potential is essentially a cosine and therefore symmetric. For moderate to high values of β the potential is strongly asymmetric and exhibits flat parts with a small negative slope with weak noise intensity [cf. Fig. 1(b)]. As an example we will consider $\beta = 5$ as shown in Fig. 1(b) (here $A(\beta = 5) \approx 3.7799$).

A simulation of the model leads to a finite transport to the right [Fig. 2(b)], however, with much smaller spreading than for the motion in the symmetric potential. The asymmetry of the potential induces a high *transport coherence*. Given a sufficiently large mean distance $\langle x_e - x_0 \rangle$ the standard deviation between different realizations can be expressed by the diffusion coefficient $D_{\text{eff},x} = \lim_{t \rightarrow \infty} \langle x^2 - \langle x \rangle^2 \rangle / 2t$ as follows:

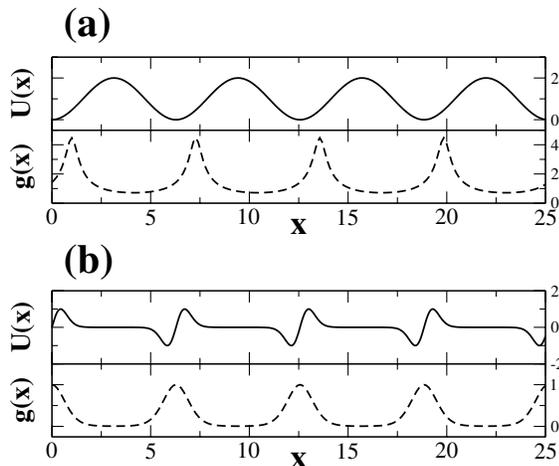


FIG. 1. Potentials and noise modulation functions discussed in this paper.

$$\sqrt{\langle x^2 - \langle x \rangle^2 \rangle} \approx \sqrt{2D_{\text{eff},x}T} \approx \sqrt{\frac{2L_x \langle x_e - x_0 \rangle}{\text{Pe}_x}}. \quad (7)$$

Here T is the mean time needed for the passage from x_0 to x_e . The ratio of the velocity $v_x = \langle x_e - x_0 \rangle / T$ and the diffusion coefficient appearing in the latter terms is the Péclet number [2,10,11] given by

$$\text{Pe}_x = \frac{v_x L_x}{D_{\text{eff},x}}. \quad (8)$$

For a fixed distance as we have used in Figs. 2(a) and 2(b), minimal spreading is associated with maximal Péclet number according to Eq. (7), hence, Pe_x is a measure of transport coherence. If on the other hand simulation time is fixed, the spreading is, per definition of $D_{\text{eff},x}$, minimized at the minimal diffusion coefficient.

Theory.—Recently, the analytical expression of the diffusion coefficient for the case of biased periodic potentials and additive noise (i.e., $g(x) = \text{const}$) has been derived in [11,12]. A calculation for multiplicative noise can be carried out using the nonlinear transformation to a system with *additive* noise [7]. For the new variable

$$y(x) = \int_0^x \frac{dz}{g(z)} \quad (9)$$

the dynamics Eq. (1) is transformed to

$$\dot{y} = \dot{x}/g(x) = -\frac{d\Phi(y)}{dy} + \sqrt{2D}\xi(t). \quad (10)$$

Here we have introduced an effective potential

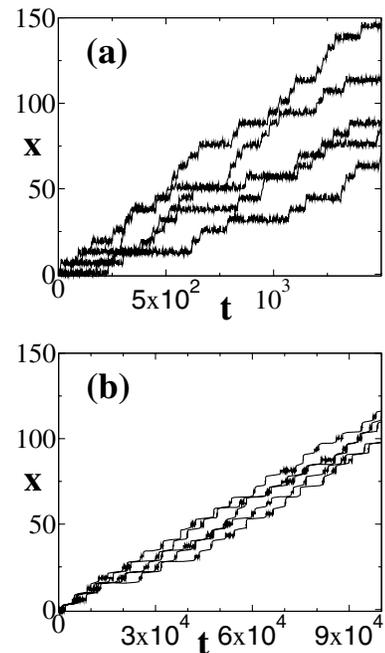


FIG. 2. Sample trajectories for $D = 0.3$ for the symmetric potential Eq. (5) (a) and the asymmetric potential Eq. (6) (b).

$$\Phi(y) = - \int_0^y d\tilde{y} \frac{f[x(\tilde{y})]}{g[x(\tilde{y})]} = \Psi[x(y)], \quad (11)$$

which can be expressed by the effective potential $\Psi(x)$ from Eq. (3) and the inverse $x(y)$ of the transformation Eq. (9). It can be readily shown that if $f(x)$ and $g(x)$ are periodic functions of period L_x , then $\Phi(y)$ will be a biased periodic potential with periods $L_y = \int_0^{L_x} dz/g(z)$. Therefore, the transformed dynamics Eq. (10) describes Brownian motion in an inclined periodic potential with *constant* noise intensity.

The underlying discrete processes which count by how many period lengths the particle has traveled are given by

$$x(t) \in \{[n_x(t) - 1]L_x, n_x(t)L_x\}, \quad (12)$$

$$y(t) \in \{[n_y(t) - 1]L_y, n_y(t)L_y\}. \quad (13)$$

It is readily seen that these processes are identical for the original and the transformed dynamics $n_x(t) \equiv n_y(t)$. Both processes determine the asymptotic mean and variance of x and y , respectively, and thus the velocity and the diffusion coefficient by $v_{x,y} = L_{x,y}n_{x,y}(t)/t$ and $D_{\text{eff},x,y} = L_{x,y}^2 \langle \Delta n_{x,y}(t) \rangle / 2t$. From these relations we find

$$v_x = v_y \frac{L_x}{L_y}, \quad D_{\text{eff},x} = D_{\text{eff},y} \left(\frac{L_x}{L_y} \right)^2, \quad \text{Pe}_x = \text{Pe}_y. \quad (14)$$

Since quadrature formulas for v_y , $D_{\text{eff},y}$, and Pe_y are known [11,12] we can also determine these quantities for the original multiplicative dynamics. Using the compact formulas from [12] we obtain after a convenient change of variables in the integrals the following expressions:

$$v_x = \frac{L_x(1 - e^{\Psi(L_x)/D})}{\int_0^{L_x} dx I_+(x)/g(x)}, \quad (15)$$

$$D_{\text{eff},x} = DL_x^2 \frac{\int_0^{L_x} dx I_+^2(x) I_-(x)/g(x)}{[\int_0^{L_x} dx I_+(x)/g(x)]^3}, \quad (16)$$

and the Péclet number, according to Eq. (8). In (15) and (16) we have abbreviated

$$I_{\pm}(x) = \pm \frac{e^{\mp \Psi(x)/D}}{D} \int_x^{x \pm L_x} dy \frac{e^{\pm \Psi(y)/D}}{g(y)}. \quad (17)$$

While the mean velocity has been previously derived (see, e.g., Ref. [6]), the expressions for the effective diffusion coefficient are to the best of our knowledge new.

For the symmetric system [Eq. (5)], an analytical expression for the effective potential (3) has been given by Büttiker [6]. For the system Eq. (6) the effective potential can also be calculated

$$\Psi(x) = A(\beta) \left[\sin(x) + \frac{1}{2} \beta \sin(x) \cos(x) - \frac{1}{2} \beta x \right]. \quad (18)$$

Using these effective potentials, we may numerically

determine the quantities of interest by means of the quadrature formulas. Since the strength of noise controls the relevant time scales of the system, we discuss in the following v_x , $D_{\text{eff},x}$, and Pe_x as functions of the noise intensity.

Results for symmetric potential.—As shown in Fig. 3(a) velocity and effective diffusion coefficient increase monotonously with growing noise; velocity saturates in the strong noise limit, whereas the diffusion coefficient behaves like $D_{\text{eff}} \sim D$ for large D . Remarkably [cf. Fig. 3(a), bottom panel], there is a noise level where the Péclet number attains a maximum versus noise intensity indicating most coherent transport. At weak noise, we find that $\text{Pe} \rightarrow 2$, which corresponds to a rare-event statistics with transitions only occurring in the direction of the slope with stronger noise (one-sided random walk). With increasing noise, escapes become faster and the time scale of relaxation (“sliding down” the slope with low noise) comes into play yielding a slightly more regular process than a simple random walk (increase and maximum in Pe). For too large noise, however, the ratcheting mechanism is weakened, backward transitions (along the low noise slope) occur, and thus, the Péclet number has to drop. Varying the other parameters (α, ϕ), we find numerically that the Péclet number is always below 3.

Results for asymmetric potential.—For the potential shown in Fig. 1(b) we find a much lower velocity [Fig. 3(b), upper panel] than for the symmetric potential. More important is the increase in regularity which manifests itself by a large maximum in the Péclet number. Furthermore, we find that the diffusion coefficient now *passes through a minimum* as a function of noise intensity. The minimum is attained in the range where the Péclet number becomes maximal and, obviously, supports the strong increase of the Péclet number. In addition, it is seen from the upper panel in Fig. 3(b) that at

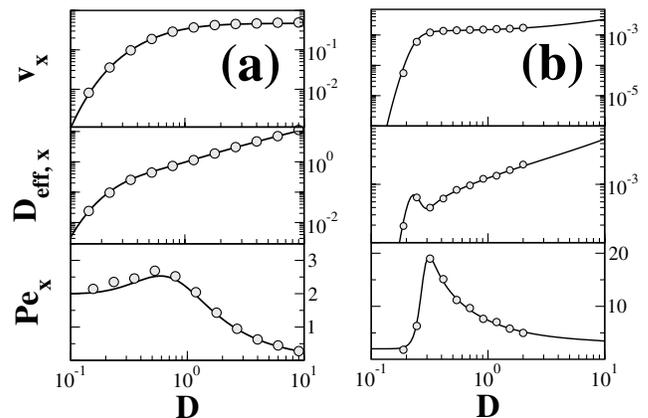


FIG. 3. Velocity, diffusion coefficient, and Péclet number vs noise intensity obtained from Eqs. (8), (15), and (16) compared to results of numerical simulations (circles). Symmetric system Eq. (5) in (a); asymmetric system Eq. (6) in (b).

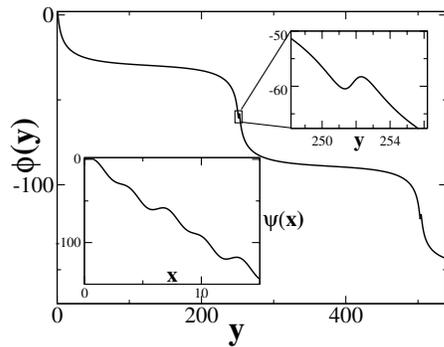


FIG. 4. Effective potential $\Psi(y)$ of the transformed dynamics Eq. (10) for the asymmetric potential, numerically determined from Eq. (11). Inset: expanded view of the potential barrier.

optimal noise intensity, the velocity is not much below its maximal value. Hence, maximizing the coherence of transport does not imply appreciable decrease in transport velocity.

The high coherence can be understood by considering the two distinct regions of motion within the potential Eq. (6). During the passage along the flat slope the particle is subject to small fluctuations [$g(x) \ll 1$], hence, the variability between realizations will be small. In contrast, the passage over the steep barrier is facilitated by a high noise level. It will be highly irregular, but fast. The combined process is a sequence of long lasting regular passages interrupted by fast escapes over the steep slopes. This process can be very regular if the passage time along the flat slope is on average large compared to the escape time over the steep slope. This separation of time scales is realized by the asymmetry of the potential *and* by the specific phase shift between potential and noise modulation: if noise intensity is larger at the flat slope, coherence of transport will be poor.

It is instructive to take a look at the potential $\Phi(y)$ of the transformed dynamics with only additive noise Eq. (10) (Fig. 4). Besides the effective bias that causes directed transport, it reveals a particular shape that explains the aforementioned time scale separation: there exist regions with rather flat negative slopes that introduce a regular large passage time and there are regions of steep slopes with a tiny barrier in the middle (magnified in the right inset) that contribute only by a small passage time. It is worth mentioning that the effective potential of the original dynamics Eq. (3) (lower left inset of Fig. 4), which is commonly discussed in systems with multiplicative noise, does not show these distinct regions and its shape might be *quite misleading* without further consideration of the multiplicative noise.

Summary.—We have found an enhancement of transport coherence for a ratchet system driven by multiplicative white noise. The basic mechanism of this enhancement is rather simple: in regions where the particle moves quasideterministically downhill the noise is de-

creased; in regions where an escape over a barrier is needed, the noise intensity is strongly increased such that the escape is fast [cf. Fig. 1(b)]. The asymmetry of the potential serves to separate the orders of magnitude of the involved time scales. If the regular downhill motion takes on average a much longer time than the irregular escape time versus the slope with strong noise, we obtain a large transport coherence. This is the case if the steeper slope is within the region of high noise intensity, whereas fluctuations are low at the flat slope.

Regarding the maximization of transport coherence by tuning the noise intensity we note a close relation of this phenomenon to *coherence resonance* (CR) observed in noisy excitable systems [13]: the excitation sequence (spike train) is most regular at a finite optimal strength of input noise. One of the measures of CR, the relative standard deviation of interspike intervals [13], is proportional to the inverse Péclet number [11] and becomes minimal (indication of CR) if the Péclet number attains its maximum (maximum of transport coherence). As consequences of coherence resonance, the excitable system exhibits a noise-induced eigenfrequency and may function as a stochastic resonator. The physical implication of maximized transport coherence discussed here, however, is entirely different: a dramatic increase in the certainty of noise-induced transport over *finite* distances.

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