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Diffusion of particles subject to nonlinear friction and a colored noise

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Abstract. Stochastic motion with nonlinear friction has several applications in physics and biology. For only a few cases, expressions for the diffusion coefficient with nonlinear friction and white Gaussian noise have been derived. Here I study one-dimensional nonlinear velocity dynamics driven by exponentially correlated ('colored') noise. For the case of a dichotomous colored noise, I calculate an exact quadrature expression for the diffusion coefficient of the resulting motion for a general odd friction function and evaluate it for three different friction functions: a pure cubic friction, a quintic friction and the Rayleigh–Helmholtz (RH) friction function. For quintic friction as well as for RH friction, the diffusion coefficient attains a minimum at a finite correlation time of the dichotomous noise. The very same effect is seen in numerical simulations with an Ornstein–Uhlenbeck process (OUP) instead of a discrete noise.

Contents

1. Introduction

The classical theory of Brownian motion builds on the Langevin equation of the form

$$\dot{x} = v, \quad \dot{v} = -\gamma v + \sqrt{2\gamma k_B T}\,\xi(t). \tag{1}$$

This describes a particle with unit mass subject to Stokes friction and a stochastic forcing; here γ is the friction coefficient, k_B is the Boltzmann constant and $\xi(t)$ is Gaussian white noise. In deriving this simple equation, a number of approximations must be made such that the friction force depends only linearly on the velocity.

The more general equations

$$\dot{x} = v, \quad \dot{v} = f(v) + g(v)\xi(t) \tag{2}$$

can describe situations where friction and also noise intensity depend in a nonlinear manner on speed. This arises not only if one goes beyond the Stokes friction approximation [1, 2] but also for relativistic Brownian motion obeying a Langevin equation that is consistent with special relativity [3, 4]. Furthermore, beyond equilibrium physics, Langevin models with nonlinear friction have found applications in the description of active motion in biology, ranging from intracellular transport by assemblies of molecular motors [5] to particles with energy depots [6–8] and crawling cells [9, 10].

The central statistics of this nonlinear Brownian motion are the mean velocity of particles

$$\langle v \rangle = \lim_{t\to\infty} \frac{\langle x(t) - x(0)\rangle}{t} \tag{3}$$

and their effective diffusion coefficient

$$D_{\text{eff}} = \lim_{t\to\infty} \frac{\langle (x(t) - \langle x(t)\rangle)^2\rangle}{2t}. \tag{4}$$

The latter coefficient quantifies the asymptotic growth of the mean-square displacement around the expected mean which moves in the steady state with mean velocity $\langle v \rangle$. The diffusion coefficient D_{eff} determines whether regular or diffusive transport dominates over a certain length- or timescale. It is thus an important measure of the output noise of nonlinear dynamics: for a general stochastic process $v(t)$ (not being necessarily a velocity), D_{eff} is also called the *noise intensity* of $v(t)$. While for equilibrium systems the diffusion coefficient is related to the linear response of the particle's mobility via a fluctuation-dissipation theorem [11], in the general nonequilibrium case of equation (2) it is unrelated to the mobility and its calculation is more challenging.

In the tractable one-dimensional (1D) case, one can calculate the mean velocity for equation (2) by standard Fokker–Planck methods. Quadrature expressions for the diffusion coefficient in a spatially symmetric system [2] and in the more general asymmetric situation [12] have been given only recently. In particular, in the nonequilibrium applications to biology, the diffusion coefficient can show a nontrivial dependence on noise intensity [2] and external forces [12]. Even in simpler situations, e.g. with additive noise [$g(v) \equiv$ const] and a simple friction law like, for instance, a power law $f(v) \sim v^n$, the diffusion coefficient reveals unexpected features. For instance, for a pure cubic friction, the diffusion coefficient does not depend on the intensity of the driving white noise fluctuations at all [13].

In the works on nonlinear friction, solely white Gaussian noise has been used. Especially for the nonequilibrium setting of the aforementioned biological problems, this may not always be a valid assumption. Driving fluctuations may be correlated, and thus it is of general interest how the features of diffusion change if the noise is colored. From a theoretical point of view, the colored noise introduces a new timescale (its correlation time) in the problem, and so it is of interest how the diffusive behavior in the system changes upon variation of this timescale.

In this paper, I study nonlinear velocity dynamics driven by a symmetric exponentially correlated noise $\eta(t)$:

$$\dot{x} = v, \quad \dot{v} = f(v) + \eta(t). \tag{5}$$

My main focus is on an analytically tractable case where $\eta(t)$ is a symmetric dichotomous noise (also called a telegraph process), which switches between $\eta = -A$ and $\eta = A$ back and forth with rate r. This noise obeys an exponential correlation function $\langle \eta(t)\eta(t+\tau)\rangle = A^2 \exp[-2r|\tau|]$ (corresponding to a Lorentzian power spectrum $S(\omega) = 4rA^2/[4r^2 + \omega^2]$), possesses variance A^2, correlation time $(2r)^{-1}$ and noise intensity $D_{\mathrm{dicho}} = A^2/(2r)$ [14, 15]. I will also study by simulations equation (5) with an Ornstein–Uhlenbeck process (OUP), a Gaussian noise with the same variance, correlation function and power spectrum as the dichotomous noise.

For the analytical calculations and the numerical examples, I assume the function $f(v)$ to be odd [$f(v) = -f(-v)$] and to yield a unique symmetric stationary velocity distribution with finite variance. Under these circumstances, the mean velocity is zero and the remaining statistic of interest is the diffusion coefficient.

According to the Kubo relation, the diffusion coefficient is given by the integral over the velocity's correlation function or, equivalently, by the velocity's power spectrum at zero frequency:

$$\begin{aligned} D_{\mathrm{eff}} &= \int_0^\infty \mathrm{d}\tau [\langle v(t)v(t+\tau)\rangle - \langle v(t)\rangle^2] \\ &= \frac{1}{2} S_{vv}(\omega = 0). \end{aligned} \tag{6}$$

There is a simple alternative interpretation of the integral formula: the integral over the normalized autocorrelation function can be regarded as the correlation time $\tau_{\rm corr}$ of the velocity process. The simple formula for the diffusion coefficient in terms of $\tau_{\rm corr}$ and the variance $\langle \Delta v^2 \rangle = \langle (v - \langle v \rangle)^2 \rangle$ then reads

$$\begin{aligned} D_{\rm eff} &= \langle \Delta v^2 \rangle \int_0^\infty {\rm d}\tau \frac{\langle v(t) v(t+\tau)\rangle - \langle v(t)\rangle^2}{\langle \Delta v^2 \rangle} \\ &= \langle \Delta v^2 \rangle \tau_{\rm corr}. \end{aligned} \tag{7}$$

From this formula it becomes evident that the strength of spatial diffusion is not solely set by the variance of velocity fluctuations alone but that their correlation time has equal importance.

Based on equation (6), I derive from the Fokker–Planck–Master equations an exact analytical quadrature expression for the diffusion coefficient. This result adds one more to the list of the exact expressions for 1D stochastic systems driven by colored dichotomous noise [14, 16–20].

I start with a simple linear case where the diffusion coefficient can be calculated by standard methods. After this, I present the derivation of the diffusion coefficient for a general odd friction function $f(v)$. I illustrate the analytical result for two simple power-law friction functions $f(v) = -\gamma v^n$ (with $n = 3$ or 5) and for the so-called Rayleigh–Helmholtz (RH) friction $f(v) = -\gamma(v^3 - v)$, which leads to bistable velocity dynamics. For a strong power-law friction ($n = 5$) and for the RH friction, I find a nonmonotonic behavior of the diffusion coefficient as a function of switching rate r. This result is also found by numerical simulations if the driving process is the OUP and not the telegraph noise. I conclude with a summary and the discussion of possible further applications and generalizations of the result for the diffusion coefficient.

2. Linear case

For the general formulae in this paper, it is useful to have a simple case to compare with. Consider the linear dynamics with $f(v) = -\gamma v$:

$$\dot{x} = v, \qquad \dot{v} = -\gamma v + \eta(t). \tag{8}$$

By Rice's method, the power spectrum of v in terms of that of η is easily obtained as

$$S_{vv}(\omega) = \frac{S_{\eta,\eta}}{\gamma^2 + \omega^2} = \frac{4rA^2}{(4r^2 + \omega^2)(\gamma^2 + \omega^2)}. \tag{9}$$

From the zero-frequency limit of the spectrum (cf equation (6)), the diffusion coefficient is given by

$$D_{\rm eff,lin} = \frac{1}{2} S_{vv}(\omega = 0) = \frac{A^2}{2r\gamma^2} = \frac{D_{\rm dicho}}{\gamma^2}. \tag{10}$$

Not surprisingly, the last equality looks similar to the usual relation between noise intensity and spatial diffusion coefficient as one finds it for the case of white noise, i.e. for the standard model of Brownian motion.

3. Derivation of the diffusion coefficient for a nonlinear friction function

For a general odd nonlinear friction function, one first has to discuss the support of the system. Consistent with the assumptions about uniqueness, finite variance and symmetry of the stationary solution, I assume that there is a maximum value v_{m} of the velocity obeying

$$f(v_{\mathrm{m}}) = -A, \quad f(-v_{\mathrm{m}}) = A \quad \text{with} \quad f'(\pm v_{\mathrm{m}}) < 0 \tag{11}$$

(A is the amplitude of the dichotomous noise) and that there are no other points within $[-v_{\mathrm{m}}, v_{\mathrm{m}}]$ where the total force on the particle vanishes. Put differently, v_{m} and $-v_{\mathrm{m}}$ are the only stable fixed points of the velocity dynamics for the two states of the dichotomous noise, respectively. Hence $[-v_{\mathrm{m}}, v_{\mathrm{m}}]$ forms the support of the velocity; v_{m} has to be found by solving equation (11). For more complicated combinations of fixed points in the context of the steady-state probability density, see [20, 21].

The master equation for the process reads

$$\partial_t p_+ = -\partial_v [f(v) + A] p_+ - r p_+ + r p_-, \tag{12}$$

$$\partial_t p_- = -\partial_v [f(v) - A] p_- + r p_+ - r p_-, \tag{13}$$

where $p_\pm$ is the probability that the velocity is v and that the dichotomous noise is in the state $\pm A$. Solving these equations for the two different initial conditions $\eta(t=0) = \pm A$ would yield the specific transition probabilities

$$\begin{aligned}
p_{++}(v, \tau; v_0) &= P(v, t+\tau, \ \eta = A | v_0, t, \eta = A), \\
p_{+-}(v, \tau; v_0) &= P(v, t+\tau, \ \eta = A | v_0, t, \eta = -A), \\
p_{-+}(v, \tau; v_0) &= P(v, t+\tau, \ \eta = -A | v_0, t, \eta = A), \\
p_{--}(v, \tau; v_0) &= P(v, t+\tau, \ \eta = -A | v_0, t, \eta = -A).
\end{aligned} \tag{14}$$

The velocity auto-correlation function, which appears in the Kubo formula for the diffusion coefficient equation (6), can be expressed by the transition probabilities $p_{\pm,\pm}$ and the stationary probabilities $P^0(v, \pm A) = p^0_\pm(v) = \lim_{\tau\to\infty} p_{\pm,\pm}(v, \tau; v_0)$, as follows:

$$\begin{aligned}
\langle v(t) v(t+\tau) \rangle &= \sum_{\eta_i, \eta_j \in \{-A, A\}} \int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v \int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v_0 \, v v_0 P(v, t+\tau, \eta_j | v_0, t, \eta_i) P^0(v_0, \eta_i) \\
&= \int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v \, v \left[\int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v_0 \, v_0 \left\{ (p_{++} + p_{-+}) p^0_+ + (p_{+-} + p_{--}) p^0_- \right\} \right].
\end{aligned} \tag{15}$$

The stationary probability densities $p^0_\pm$ are known [14, 22]: their sum $p^0(v) = p^0_+(v) + p^0_-(v)$ and difference $q^0(v) = p^0_+(v) - p^0_-(v)$ obey the ordinary differential equations

$$(f(v) p^0)' = -A (q^0)', \tag{16}$$

$$(f(v) q^0)' - 2 r q^0 = -A (p^0)', \tag{17}$$

where the prime denotes the derivative with respect to v. The explicit solution for $p^0(v)$ needed below reads [14, 22]

$$p^0(v) = \frac{\exp[-U(v)]}{(A^2 - f^2(v)) \int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}y \, \mathrm{e}^{-U(y)} / (A^2 - f^2(y))}, \tag{18}$$

where the function $U(v)$ reads

$$U(v) = -2r \int_0^v \mathrm{d}x \frac{f(x)}{A^2 - f^2(x)}. \tag{19}$$

I note that, because of the assumed properties of the function $f(v)$, both the potential and the probability density are even functions of v and the steady-state solution possesses vanishing mean and a finite variance. Further, as discussed, for instance in [14, 20], the probability density has integrable divergences at the boundaries for sufficiently low switching rate

$$\lim_{v \to \pm v_\mathrm{m}} p(v) = \infty \quad \text{for} \quad r < |f'(v_\mathrm{m})| \tag{20}$$

but vanishes for higher rates at v_m. The divergence at low rates reflects a pronounced bimodality of the velocity density.

The four transition probabilities $p_{\pm,\pm}$ appearing in equation (15) are not feasible for an arbitrary nonlinear function $f(v)$. However, full knowledge of $p_{\pm,\pm}(v)$ is not strictly required—all one has to know is the time integral from zero to infinity over the transition probabilities. For this time integral one obtains a number of ordinary differential equations that can be solved by quadratures.

First, I write the diffusion coefficient as

$$D_\mathrm{eff} = \int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}v\, p(v) v, \tag{21}$$

where $p(v)$ is defined by

$$\begin{aligned} p(v) &= \int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}v_0 \int_0^\infty \mathrm{d}\tau\, v_0 \left\{ (p_{++} + p_{-+}) p_+^0 + (p_{+-} + p_{--}) p_-^0 \right\} \\ &= \hat{p}_{++}(v) + \hat{p}_{-+}(v) + \hat{p}_{+-}(v) + \hat{p}_{--}(v), \end{aligned} \tag{22}$$

where the terms in the second line correspond to the respective double integrals in the first line.

It turns out that one still needs another function $q(v)$ defined by

$$\begin{aligned} q(v) &= \int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}v_0 \int_0^\infty \mathrm{d}\tau\, v_0 \left\{ (p_{++} - p_{-+}) p_+^0 + (p_{+-} - p_{--}) p_-^0 \right\} \\ &= \hat{p}_{++}(v) - \hat{p}_{-+}(v) + \hat{p}_{+-}(v) - \hat{p}_{--}(v). \end{aligned} \tag{23}$$

Next, I integrate the master equations for the two different sets of initial conditions $[p_+(\tau = 0) = 1, p_-(\tau = 0) = 0$ or $p_+(\tau = 0) = 0, p_-(\tau = 0) = 1]$ over time and after multiplication with $p_\pm^0(v_0) v_0$ also over the initial velocity v_0. Taking into account on the left-hand sides the values for $t = 0$ (initial conditions) and for $t \to \infty$ (the stationary probability densities), one obtains for the single terms $\hat{p}_{\pm,\pm}(v)$

$$\langle v \rangle_+ p_+^0 - v p_+^0 = -\partial_v [f(v) + A] \hat{p}_{++} - r \hat{p}_{++} + r \hat{p}_{-+}, \tag{24}$$

$$\langle v \rangle_+ p_-^0 = -\partial_v [f(v) - A] \hat{p}_{-+} + r \hat{p}_{++} - r \hat{p}_{-+}, \tag{25}$$

$$\langle v \rangle_- p_+^0 = -\partial_v [f(v) + A] \hat{p}_{+-} - r \hat{p}_{+-} + r \hat{p}_{--}, \tag{26}$$

$$\langle v \rangle_- p_-^0 - v p_-^0 = -\partial_v [f(v) - A] \hat{p}_{--} + r \hat{p}_{+-} - r \hat{p}_{--}. \tag{27}$$

For symmetry reasons, the stationary velocity is zero ($\langle v\rangle = \langle v\rangle_+ + \langle v\rangle_- = 0$) and the respective terms on the left-hand sides involving $\langle v\rangle_\pm$ vanish in the equations for p and q. Adding all the equations yields a first ordinary differential equation for the two functions $p(v)$ and $q(v)$:

$$(f(v)p(v))' + Aq'(v) = v(p_+^0(v) + p_-^0(v)) = vp^0(v), \tag{28}$$

where the prime denotes differentiation with respect to v and I have used the stationary marginal probability density $p^0(v)$ given in equation (18). All transition probabilities (and their temporal integrals) have to vanish beyond the support of v. The formal solution $q(v)$ of the above differential equation consistent with this condition is

$$q(v) = \frac{1}{A}\left[-f(v)p(v) + \int_{-v_\mathrm{m}}^{v} \mathrm{d}v_1\, v_1 p^0(v_1)\right] \tag{29}$$

(for $v \to v_\mathrm{m}$, the integral term approaches the stationary mean velocity, which is zero).

Taking the difference between the sums of equations (24) and (26), on the one hand, and equations (25) and (27), on the other hand, yields the second ordinary differential equation

$$(f(v)q(v))' + Ap'(v) + 2rq(v) = v(p_+^0(v) - p_-^0(v)) = vq^0(v), \tag{30}$$

where I have used the stationary difference $q^0(v) = p_+^0(v) - p_-^0(v)$. If one inserts the formal solution equation (29), one obtains the following inhomogeneous first-order equation for the desired function $p(v)$:

$$([A^2 - f^2(v)]p(v))' - 2rf(v)p(v) = -[2r + f'(v)]\int_{-v_\mathrm{m}}^{v} \mathrm{d}v_1\, v_1 p^0(v_1) + v[Aq^0(v) - f(v)p^0(v)]. \tag{31}$$

Using the first differential equation for p^0 and q^0, equation (16), together with the boundary condition that both vanish outside $[-v_\mathrm{m}, v_\mathrm{m}]$, one has $Aq^0(v) = -f(v)p^0(v)$ and can thus express the right-hand side of equation (31) solely in terms of $p^0(v)$. The remaining equation for $p(v)$ can be solved by standard methods (it is similar to the solution of the stationary problem [14, 22] with an additional inhomogeneity). The solution that satisfies natural boundary conditions reads

$$p(v) = -\frac{\exp[-U(v)]}{A^2 - f^2(v)}\int_{-v_\mathrm{m}}^{v} \mathrm{d}x\, \mathrm{e}^{U(x)}\left[2xf(x)p_0(x) + (f'(x) + 2r)\int_{-v_\mathrm{m}}^{x} \mathrm{d}y\; y\; p_0(y)\right]. \tag{32}$$

Inserting this expression into equation (21) and further simplifying the integrals as outlined in appendix A yield the main result of this paper, the quadrature formula for the effective diffusion coefficient

$$D_\mathrm{eff} = 2rA^2 \frac{\int_0^{v_\mathrm{m}} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)}\left(\int_x^{v_\mathrm{m}} \mathrm{d}y \frac{\mathrm{e}^{-U(y)}}{A^2 - f^2(y)}\, y\right)^2}{\int_0^{v_\mathrm{m}} \mathrm{d}z\, \mathrm{e}^{-U(z)}[A^2 - f^2(z)]^{-1}}. \tag{33}$$

The numerical evaluation of this formula is outlined in appendix B. In the following section, I verify this formula in three simple limiting cases.

4. Special cases

For a linear friction force $f(v) = -\gamma v$ with $v_m = A/\gamma$, the exponentials in the integrand can be calculated analytically:

$$U(x) = -2r \int_0^x \mathrm{d}y \frac{-\gamma y}{A^2 - \gamma^2 y^2} = -\frac{r}{\gamma} \ln\left(1 - \frac{\gamma^2}{A^2} x^2\right)$$
$$\Rightarrow \exp[-U(x)] = \left(1 - \frac{\gamma^2}{A^2} x^2\right)^{r/\gamma}. \tag{34}$$

Using these expressions, the quadrature expression for the diffusion coefficient can be recast into

$$D_{\mathrm{eff}} = \frac{2r}{A^2} \frac{\int_0^{v_m} \mathrm{d}x \frac{\left(\int_x^{v_m} \mathrm{d}y [1-\gamma^2 y^2/A^2]^{r/\gamma-1} y\right)^2}{(1-\gamma^2 x^2/A^2)^{1+r/\gamma}}}{\int_0^{v_m} [1-\gamma^2 z^2/A^2]^{r/\gamma-1}\, \mathrm{d}z} \tag{35}$$

$$= \frac{2r}{A^2} \frac{\int_0^{v_m} \mathrm{d}x \frac{\left(\frac{A^2}{2r\gamma}[1-\gamma^2 x^2/A^2]^{r/\gamma}\right)^2}{(1-\gamma^2 x^2/A^2)^{1+r/\gamma}}}{\int_0^{v_m} [1-\gamma^2 z^2/A^2]^{r/\gamma-1}\, \mathrm{d}z} = \frac{A^2}{2r\gamma^2}, \tag{36}$$

which agrees with the result equation (10) from section 2.

One can also perform the well-known limit to the white-noise case with $A \to \infty$, $r \to \infty$ with $D = A^2/(2r) = \mathrm{const}$. In both the potential $U(x)$ and the integrand of equation (33), the factor A^2 can be pulled out of the difference $A^2 - f^2(x)$. In the limit $A \to \infty$, the resulting term f^2/A^2 approaches zero and the potential can be written as

$$U(x) = -\frac{2r}{A^2} \int_0^x \mathrm{d}y \frac{f(y)}{1 - f^2(y)/A^2} \to \frac{V(x)}{D}, \tag{37}$$

where

$$V(x) = -\int_0^x \mathrm{d}y\, f(y), \tag{38}$$

i.e. in terms of the common potential defined by the integrated negative drift term in the Langevin equation. With the same limit in all integrals of equation (33), the diffusion coefficient attains the form

$$D_{\mathrm{eff}} = \frac{1}{D} \frac{\int_0^\infty \mathrm{d}x e^{V(x)/D} \left(\int_x^\infty \mathrm{d}y e^{-V(y)/D} y\right)^2}{\int_0^\infty \mathrm{d}z e^{-V(z)/D}}, \quad \text{for } r, A \to \infty \quad \text{with} \quad D = \frac{A^2}{2r} = \mathrm{const}, \tag{39}$$

which coincides with the result in [2] for the special case of an additive white Gaussian noise.

For a general nonlinear friction function, equation (33) has to be evaluated numerically. In particular at small rate, this numerical integration becomes difficult close to the boundary $v \approx v_m$ because here the term $A^2 - f^2(v)$ approaches zero and the integrands in the potential and in the quadratures diverge. In the appendix I derive a formula in which the critical parts of the integration in a neighborhood of v_m are performed analytically. This formula allows one to also calculate the limit of the weak rate of dichotomous noise analytically (see appendix B), yielding

$$D_{\mathrm{eff}} = \frac{v_m^2}{2r}, \quad \text{for } r \to 0 \quad \text{with} \quad A = \mathrm{const}. \tag{40}$$

This result makes perfect sense. At weak input rate, the velocity has time to relax to a value close to v_m and to stay a long time at $v = \pm v_m$. Thus, the velocity process itself becomes essentially a dichotomous noise (enslaved by the slow driving). For a velocity obeying a telegraph noise with amplitude v_m, the diffusion coefficient is known for a long time [16] and is exactly given by equation (40).

5. Two examples for a power-law friction

I discuss the analytical result for a specific friction function $f(v)$ following a power law. The stochastic differential equation reads

$$\dot{x} = v, \quad \dot{v} = -\gamma v^n + \eta(t), \tag{41}$$

where I will consider the cases where $n = 3$ (cubic friction) and $n = 5$ (quintic friction). For a power-law friction, one can simplify the expression for the diffusion coefficient by choosing new integration variables $z = x\sqrt[n]{A/\gamma}$ in the integrals appearing in the potential $U(x)$ in equation (19) as well as in the integrals of equation (33). The parametric dependence on A and γ can then be lumped into a rescaled rate and a prefactor, as follows:

$$D_{\text{eff}}(r, A, \gamma) = \frac{D_{\text{eff}}\left(r(A^{n-1}\gamma)^{-1/n}, 1, 1\right)}{(A^{n-3}\gamma)^{1/n}}. \tag{42}$$

This result may also be derived by scaling arguments (renormalization theory) similar to the power-law friction in the white-noise case [13].

From equation (42), one can conclude that the diffusion coefficient for $n = 3$ (cubic friction) depends on the amplitude only via the rescaled rate $\tilde{r} = r(A^{n-1}\gamma)^{-1/n}$. For arbitrary n, the relation in equation (42) gives us a means to obtain the diffusion coefficient as a function of switching rate for any amplitude A once we obtain it for one specific amplitude. For clarity, I will first, however, discuss diffusion coefficients for $n = 3$ and 5 for various amplitudes and then illustrate equation (42) by rescaling the axes.

In figure 1, I show the diffusion coefficient as a function of the rate of dichotomous noise. For cubic friction, the diffusion coefficient is a monotonically decreasing function of the rate. At small rate, D_{eff} decays in proportion to r^{-1} in accordance with the limit in equation (40). In the opposite limit of high switching rate, one observes a saturation at the level found for the white-noise case. I recall the result from [13] that for a cubic friction law and a white-noise driving, the diffusion coefficient is approximately given by

$$D_{\text{eff}} \approx 0.4875/\gamma, \quad \text{for } f(v) = -\gamma v^3 \text{ and white-noise driving} \tag{43}$$

(the approximation just relates to the numerical prefactor and not the dependence on parameters). So in the white-noise limit the diffusion coefficient does not depend on the noise intensity and hence does not depend on r and A. This is why in figure 1(a) all the curves saturate at the same level.

It is remarkable, however, that the onset of saturation differs strongly for different driving amplitudes. With growing amplitude A the onset of saturation moves to higher rates as $A^{2/3}$ (all curves can be collapsed on one curve by using a rescaled rate $\tilde{r} = r/(\gamma A^2)^{1/3}$, as done in figure 1(b)). This is due to the nonlinearity of the velocity dynamics. One can estimate the time

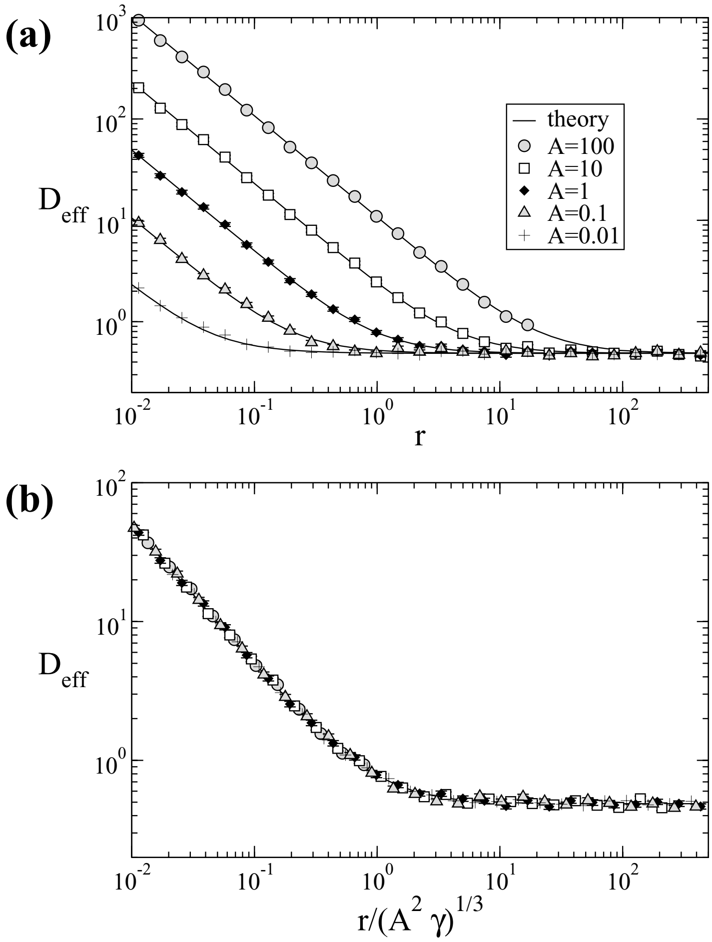

Figure 1. Diffusion coefficient for a cubic friction law $f(v) = -\gamma v^3$, shown as a function of the switching rate of dichotomous noise for various amplitudes as indicated and with $\gamma = 1$. Theory (lines) is compared to the results of simulations (symbols) of the Langevin equation equation (41). In (b), all simulation data are shown versus a rescaled rate. Simulations were performed for an ensemble of 10^3 trajectories and a time window of $T = 10^5$ using a simple stochastic Euler procedure with a time step $\Delta t = 10^{-3}$ or $\Delta t = 10^{-4}$ for very high rates.

for the velocity to relax to another steady state after a switch in the driving force has occurred. If, for example, at time t the input $\eta(t) = -A \to A$, then the time to go from a value close to $v = -v_\mathrm{m} = -\sqrt[3]{A/\gamma}$ to a value αv_m (where α is close to but smaller than one) shows the following proportionality:

$$T = \int_{-v_\mathrm{m}}^{\alpha v_\mathrm{m}} \frac{\mathrm{d}v}{A - \gamma v^3} \sim (A^2\gamma)^{-1/3}. \tag{44}$$

This implies that the stronger the driving, the faster the system relaxes to the new steady-state value after a switching of the input. The dichotomous input noise 'looks white' to the velocity dynamics if its input rate is much larger than the relaxation rate, i.e. $r \gg 1/T \sim (A^2\gamma)^{1/3}$. Incidentally, it is the same ratio of switching rate and relaxation rate that determines

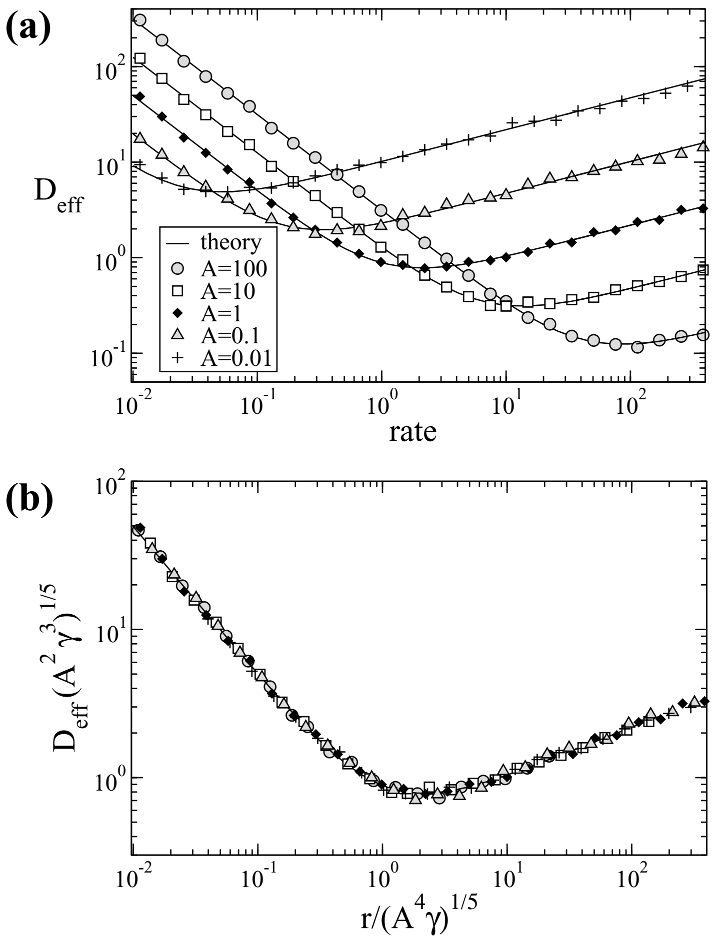

Figure 2. Diffusion coefficient for a quintic friction law $f(v) = -\gamma v^5$. In (a) the coefficient is shown as a function of the switching rate of dichotomous noise for various amplitudes, as indicated, and with $\gamma = 1$. Theory (lines) is compared to results of simulations (symbols) of the Langevin equation equation (41). Simulation parameters as in figure 1. In (b), rescaled axes have been used for which all data from (a) collapse on one curve.

whether the stationary probability density diverges or vanishes at the boundaries; according to equation (20), divergences exist only if $r < 3(A^2\gamma)^{1/3}$.

For quintic friction (figure 2), one observes a different behavior of the diffusion coefficient at high switching rate—the diffusion coefficient, after having reached a minimum, increases again. For high rates, the system approaches the white-noise limit with a noise intensity $D_{\text{dicho}} = A^2/(2r)$ that decreases with increasing rate. In [13] it was shown that the diffusion coefficient diverges for a quintic friction function with vanishing noise like

$$D_{\text{eff}} \approx 0.3737\gamma^{-2/3} D^{-1/3}, \quad \text{for } f(v) = -\gamma v^5 \text{ and white-noise driving} \tag{45}$$

(again, the approximation involves only the numerical prefactor). Thus, one can expect that at high rates the diffusion coefficient increases like $r^{1/3}$, which is indeed observed in figure 2(a). As a consequence of the decrease at small rates and the increase at high rates, one finds a minimal

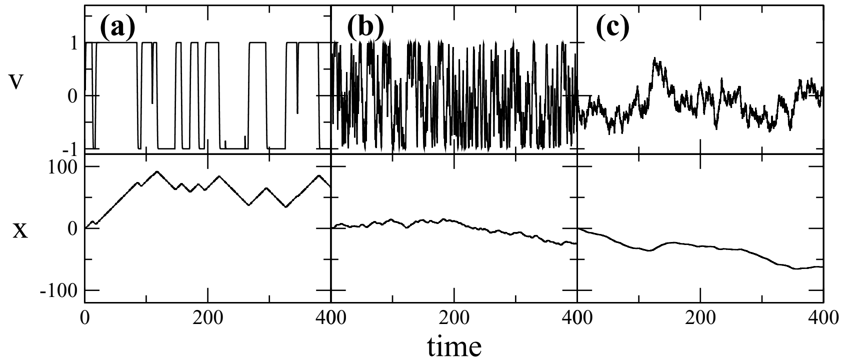

Figure 3. Velocities (upper row) and spatial coordinate (lower row) of a Brownian particle subject to quintic friction and a dichotomous noise with switching rates $r = 0.1$ (a), $r = 2$ (b) and $r = 100$ (c). The rate in (b) corresponds to the minimum of the diffusion coefficient in figure 2.

diffusion coefficient at intermediate rate. This rate (comparable to the onset of saturation for the cubic friction) depends on the amplitude (figure 2(a)).

According to equation (42), one can collapse all the curves on one if one rescales the rate and the diffusion coefficient (cf figure 2(b)). From this plot we can read off that the diffusion coefficient becomes minimal about values of rate and amplitude that satisfy

$$r_{\rm min} \approx 2(A^4\gamma)^{1/5}. \tag{46}$$

Remarkably, at this rate, divergences of the stationary probability density at $v = \pm v_{\rm m}$ are still present; they only vanish according to equation (20) at $r = 5(A^4\gamma)^{1/5} > r_{\rm min}$. Thus, at $r = r_{\rm min}$ a significant fraction of the probability is still at high speed (the standard deviation is about two-thirds the one at slow driving) but the correlation time (the second factor in the diffusion coefficient) is significantly reduced compared to the case of slow driving. The correlation time becomes large in the two limits of small and large r for different reasons. While a small rate determines the velocity's correlation time directly ($\tau_{\rm corr} \sim 1/r$), a high switching rate has a more indirect effect: it results in a small standard deviation of the velocity and thus in an extremely weak dissipation by the nonlinear friction, which, in turn, yields a long memory and thus a large correlation time. If the rate is about $r_{\rm min}$, none of these two effects is dominant and, consequently, the correlation time and the variance of the velocity are moderate and hence also their product $D_{\rm eff}$ is small.

Velocity traces and spatial trajectories for a quintic friction and three different rates are shown in figure 3. At low rate (figure 3(a)), the input dichotomous fluctuations carry over to the velocity (upper panel) and the motion is bidirectional. At intermediate rate (figure 3(b)), corresponding to the minimum in figure 2(b), the velocity fluctuations are short-correlated and also display reduced variance. In the limit of strong rate (figure 3(c)), the variance of the velocity is further reduced; its correlation time, however, is strongly increased. As outlined above, this is so because for the typical small velocity values the dissipation ('forgetfulness') is rather small because it depends on the fifth power of velocity. Hence, in this limit the motion is strongly persistent and shows for that reason a large diffusional spread.

6. Rayleigh friction with dichotomous driving

As a further example, I consider an RH friction function given by

$$f(v) = \gamma(v - v^3). \tag{47}$$

Here the friction function attains negative values at small speed (it is pumping energy into the system) while at large velocities a cubic friction dominates. It is a standard example of an active Brownian motion [6, 7, 23] that has been applied with white-noise driving, for instance, in the context of coupled molecular motors [5]. In contrast to the case of power-law friction, for the RH friction function a vanishing velocity ($v = 0$) becomes dynamically unstable in the absence of noise and two symmetric solutions corresponding to $v = -1$ and $v = 1$ appear. Small amounts of white noise then result in a bidirectional motion of the particle, where changes in direction correspond to noise-induced transitions between the two metastable velocities $v = -1$ and 1.

If driven by dichotomous noise, it is clear that one has to demand a sufficiently large amplitude of the driving that allows for transitions between positive and negative velocities. Otherwise, the condition of a unique stationary solution made above would not be met and the steady-state probability for the particle to have positive or negative velocity would depend on the initial ensemble (this is the nonergodic case of adjacent stable fixed points of the two force fields $f(v) \pm A$, briefly discussed in [21]). For the friction function equation (47), it is convenient to prescribe a stable asymptotic value $v_{\rm m}$ for which

$$f(v_{\rm m}) + A = 0 \tag{48}$$

and to find the needed amplitude

$$A = \gamma(v_{\rm m}^3 - v_{\rm m}). \tag{49}$$

Here I choose $v_{\rm m} = 1.5$, $\gamma = 1$ and obtain $A = 1.875$ (of course, for given γ and A, $v_{\rm m}$ could also be found as the solution of a cubic equation).

In figure 4, I show the results of numerical simulations compared to the evaluation of the quadrature formula for the case of the RH friction function. As for the quintic friction function, one finds in this case a minimum of the diffusion coefficient as a function of the dichotomous switching rate r. This minimum is based on the fact that the correlation time of the velocity diverges in both limits of vanishing and infinite rate. In contrast to what was found for the power-law friction, however, this divergence is based on bidirectionality of the particle's motion in *both* limiting cases. At small rate, the velocity follows adiabatically the dichotomous driving process and is therefore bidirectional. At high switching rate, the driving approaches the white-noise limit in which the motion is bidirectional because of the metastability of the RH dynamics. With increasing rate in the latter limit, the noise intensity of fluctuations decreases and the Kramers rate of transitions between the metastable velocities decreases exponentially. For this reason, the diffusion coefficient (depending on the inverse of the switching rate) increases more strongly in the limit $r \to \infty$ than in the limit $r \to 0$.

In figure 5, I show velocity traces and spatial trajectories for different switching rates. In the two extreme cases of small rate (a) and large rate (c), the velocity is clearly bidirectional with long waiting times between reversals of the velocity. For the rate at which the diffusion coefficient becomes minimal, the velocity histogram is still bimodal because of the divergences at the boundaries but close to a uniform distribution for the more typical smaller velocities (not shown). Additionally, reversals occur far more often and thus the resulting trajectory shows a slower diffusive spreading.

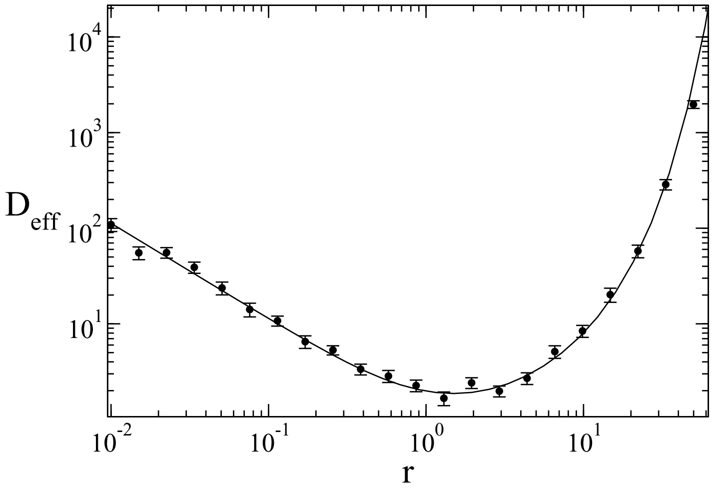

Figure 4. Diffusion coefficient for the RH friction function equation (47) versus switching rate of dichotomous noise with $\gamma = 1$. Theory (line) is compared to results of simulations (symbols) of the Langevin equation (5). For the estimation of the diffusion coefficient, 100 realizations with a time window of $T = 10^4$ and a time step of 10^{-4} were simulated.

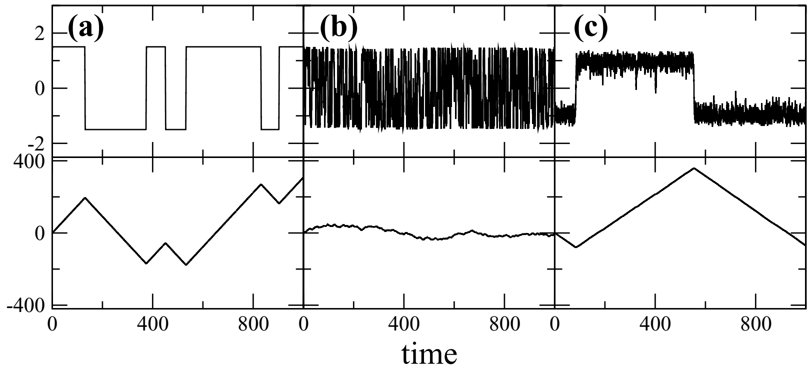

Figure 5. Velocities (upper row) and spatial coordinate (lower row) of an active Brownian particle subject to RH friction and a dichotomous noise with switching rates $r = 0.01$ (a), $r = 2$ (b) and $r = 30$ (c). The rate in (b) corresponds to the minimum of the diffusion coefficient in figure 4.

7. Nonlinear friction and Ornstein–Uhlenbeck noise

Are the minima of the diffusion coefficient versus switching rate simply due to the discrete character of the input noise? This is a valid question: minimizing the diffusion coefficient by tuning the timescale of the correlated driving is a remarkable finding that, however, would be less interesting if it hinged upon the detailed statistical features of the noise. Here I show numerically that the diffusion coefficient also passes through a minimum as a function of the decay rate if one replaces the dichotomous noise by a Gaussian noise with exponential

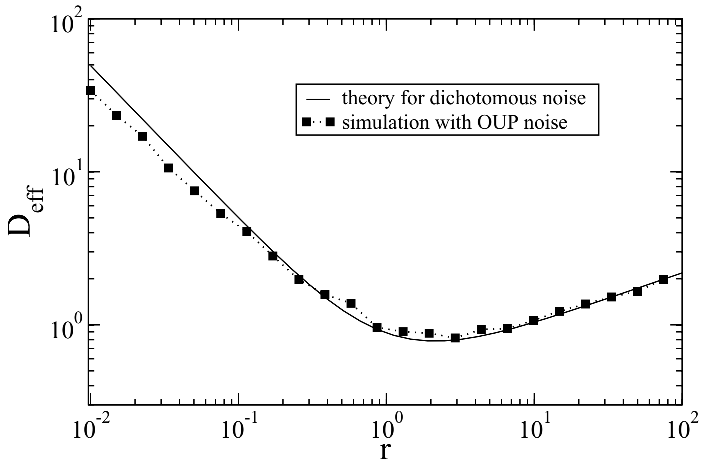

Figure 6. Diffusion coefficient versus rate for a quintic friction and an OUP noise with $A = 1$ and $\gamma = 1$.

correlation function, i.e. by an OUP. I consider now the dynamics

$$\dot{x} = v, \quad \dot{v} = f(v) + \eta(t), \tag{50}$$

$$\dot{\eta} = -2r\eta + 2A\sqrt{r}\xi(t), \tag{51}$$

where $\eta(t)$ is the OUP that is determined by a linear stochastic differential equation driven by a white Gaussian noise $\xi(t)$. The unusual scaling with A and r in equation (51) ensures that the correlation function, variance and correlation time of the OUP are the same as for the dichotomous process. For the OUP, the parameter r should be referred to as (half) the decay rate of the correlation.

The diffusion coefficient for a quintic friction function is shown in figure 6 and compared to the analytical solution for dichotomous driving with the same variance. Remarkably, the diffusion coefficient passes through a minimum as a function of the relaxation rate of the OUP, as it did for the dichotomous driving. Moreover, the diffusion coefficient for the OUP case is rather close to the one obtained for the discrete noise. In particular, the positions and values of the minima agree well. One can expect that both lines coincide in the limit of high rate because here both the OUP and the dichotomous noise approach the white noise with intensity $D_{\mathrm{wn}} = A^2/(2r)$. The good agreement of the two curves at low and moderate rates, however, is somewhat surprising.

For velocity dynamics with RH friction and OUP noise, the diffusion coefficient passes through a minimum as a function of the correlation's decay rate and is also close to the analytical result for dichotomous driving (cf figure 7). In the case of the RH friction one can, however, expect not only quantitative but also qualitative differences as soon as the amplitude of the noise becomes too small to drive transitions from one metastable velocity to the other. In the comparable case of unbounded OUP noise, there will always be a finite probability to make this transition, and thus the resulting diffusive behavior with OUP noise is expected to be quite different from the one under dichotomous stimulation (not shown).

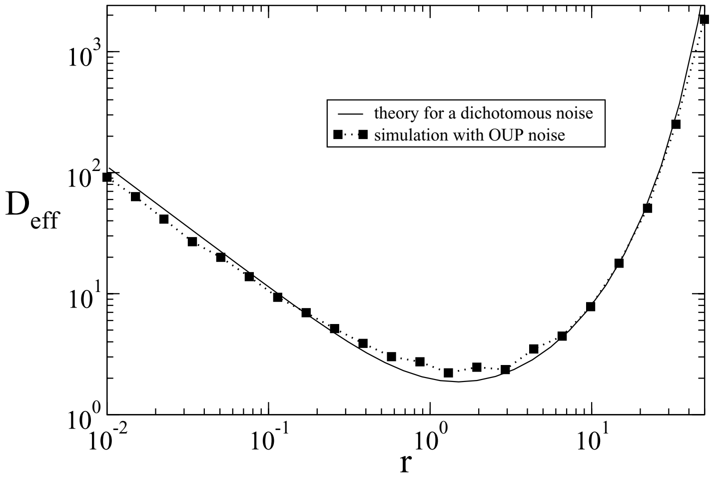

Figure 7. Diffusion coefficient versus rate for an RH friction and an OUP noise with $A = 1.875$ and $\gamma = 1$.

8. Conclusions

In this paper, I have studied the effect of colored noise on the diffusion of Brownian particles with nonlinear friction. In particular, I investigated how the diffusion coefficient depends on the decay rate of the noise correlations.

I derived an expression for a one-dimensional Brownian motion with nonlinear friction driven by a colored dichotomous noise. I have discussed this general result for the specific cases of a power-law function and an RH friction function and observed an excellent agreement between theory and simulations. I discussed specifically how, in the limiting cases of small and large switching rate of the telegraph process, qualitatively different behaviors can be expected. At slow driving a bidirectional motion corresponding to large diffusion is observed for all considered friction functions because the driving dichotomous process carries over to the velocity process. At fast driving the diffusion approaches that of a particle with nonlinear friction and a white Gaussian noise previously discussed in [13]. With a constant amplitude scaling and an exponent of power-law friction $n > 3$, I observed a minimum in the diffusion coefficient at moderate input rate, which is a simple consequence of the divergence of the diffusion coefficient in the two limit cases. A minimum is also present for an RH friction function because here the diffusion coefficient diverges in the white-noise limit due to the bistability of the velocity dynamics.

If the dichotomous noise is replaced by an Ornstein–Uhlenbeck noise, the diffusion coefficient also shows a nonmonotonic dependence on the timescale of the driving. Interestingly, for the considered numerical examples, the diffusion coefficient for the OUP case is very close to the one calculated for the dichotomous driving. Thus, at least in cases where the finite support of the discrete noise does not matter, the diffusion coefficient seems to be largely determined by the correlation function of the driving noise and by the nature of the friction function in the velocity dynamics.

The results in this paper provide another example of a nonlinear stochastic system with a nonmonotonic relation between the statistics of input and output fluctuations as, for instance, stochastic resonance [24], coherence resonance [25], resonant activation [26] and noise-induced transport [27].

Interesting analytical extensions of the present work are obvious: the case of a driving OUP has been only numerically studied here but is certainly worth some analytical efforts, too. The diffusion of a particle with nonlinear friction and dichotomous driving in two or three spatial dimensions is another analytical challenge. The most tractable but physically also very promising extension of the presented problem seems to me the case of broken spatial symmetry, which can be realized by an asymmetric dichotomous driving and/or an asymmetric friction function. These problems will be studied elsewhere.

Acknowledgment

I thank Lutz Schimansky-Geier for an inspiring discussion on the subject of this work.

Appendix A. Simplification of the integrals in the expression for the diffusion coefficient

Inserting the stationary density equation (18) into equation (32) and the latter solution into equation (21) yields

$$D_{\mathrm{eff}} = -\frac{\int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v \frac{v\mathrm{e}^{-U(v)}}{A^2-f^2(v)} \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x\, \mathrm{e}^{U(x)} \left[\frac{2xf(x)\mathrm{e}^{-U(x)}}{A^2-f^2(x)} + (f'(x)+2r) \int_{-v_{\mathrm{m}}}^{x} \mathrm{d}y\, \frac{y\mathrm{e}^{-U(y)}}{A^2-f^2(y)}\right]}{\int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}y\, \mathrm{e}^{-U(y)}/(A^2-f^2(y))}. \tag{A.1}$$

First one simplifies the integral over x involving $f'(x)$. This can be integrated by part leading to

$$\begin{aligned}
\int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x\mathrm{e}^{U(x)} f'(x) \int_{-v_{\mathrm{m}}}^{x} \mathrm{d}y\, \frac{y\mathrm{e}^{-U(y)}}{A^2-f^2(y)} &= f(v)\mathrm{e}^{U(v)} \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}y\, \frac{y\mathrm{e}^{-U(y)}}{A^2-f^2(y)} \\
&\quad +2r \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x \frac{f^2(x)\mathrm{e}^{U(x)}}{A^2-f^2(x)} \int_{-v_{\mathrm{m}}}^{x} \mathrm{d}y\, \frac{y\mathrm{e}^{-U(y)}}{A^2-f^2(y)} - \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x\, \frac{xf(x)}{A^2-f^2(x)} \\
= f(v)\mathrm{e}^{U(v)} \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}y\, \frac{y\mathrm{e}^{-U(y)}}{A^2-f^2(y)} &- \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x\, \frac{xf(x)}{A^2-f^2(x)} \\
&\quad +2A^2 r \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2-f^2(x)} \int_{-v_{\mathrm{m}}}^{x} \mathrm{d}y\, \frac{y\mathrm{e}^{-U(y)}}{A^2-f^2(y)} \\
&\quad -2r \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x\mathrm{e}^{U(x)} \int_{-v_{\mathrm{m}}}^{x} \mathrm{d}y\, \frac{y\mathrm{e}^{-U(y)}}{A^2-f^2(y)}.
\end{aligned}$$

The term in the last line will cancel with the term with prefactor $2r$ in equation (A.1). The first two terms together with the first term in the square bracket of equation (A.1) when integrated over v do not provide a contribution:

$$\begin{aligned}
&\int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v \frac{vf(v)}{A^2-f^2(v)} \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x \frac{x\,\mathrm{e}^{-U(x)}}{A^2-f^2(x)} + \int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v \frac{v\mathrm{e}^{-U(v)}}{A^2-f^2(v)} \int_{-v_{\mathrm{m}}}^{v} \mathrm{d}x \frac{xf(x)}{A^2-f^2(x)} \\
&\quad = \int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}v \frac{vf(v)}{A^2-f^2(v)} \int_{-v_{\mathrm{m}}}^{v_{\mathrm{m}}} \mathrm{d}x \frac{x\mathrm{e}^{-U(x)}}{A^2-f^2(x)} = 0,
\end{aligned} \tag{A.2}$$

where in the second term in the first line the order of integration was exchanged and the integration variables renamed; the resulting inner integral in the second line is proportional to the stationary mean value, which vanishes because of symmetry.

The only remaining term in the numerator of equation (A.1) can be further simplified by exchanging the order of integrations:

$$\begin{aligned}
&\int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}v \frac{v\mathrm{e}^{-U(v)}}{A^2 - f^2(v)} \int_{-v_\mathrm{m}}^{v} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)} \int_{-v_\mathrm{m}}^{x} \mathrm{d}y \frac{y\mathrm{e}^{-U(y)}}{A^2 - f^2(y)} \\
&\qquad = \int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)} \int_{-v_\mathrm{m}}^{x} \mathrm{d}y \frac{y\mathrm{e}^{-U(y)}}{A^2 - f^2(y)} \int_{x}^{v_\mathrm{m}} \mathrm{d}v \frac{v\mathrm{e}^{-U(v)}}{A^2 - f^2(v)} \\
&\qquad = \int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)} \left[\underline{\int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}y \frac{y\mathrm{e}^{-U(y)}}{A^2 - f^2(y)}} - \int_{x}^{v_\mathrm{m}} \mathrm{d}y \frac{y\mathrm{e}^{-U(y)}}{A^2 - f^2(y)} \right] \\
&\qquad\quad \times \int_{x}^{v_\mathrm{m}} \mathrm{d}v \frac{v\mathrm{e}^{-U(v)}}{A^2 - f^2(v)} \\
&\qquad = -\int_{-v_\mathrm{m}}^{v_\mathrm{m}} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)} \left[\int_{x}^{v_\mathrm{m}} \mathrm{d}y \frac{y\mathrm{e}^{-U(y)}}{A^2 - f^2(y)} \right]^2 \\
&\qquad = -2 \int_{0}^{v_\mathrm{m}} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)} \left[\int_{x}^{v_\mathrm{m}} \mathrm{d}y \frac{y\mathrm{e}^{-U(y)}}{A^2 - f^2(y)} \right]^2,
\end{aligned}$$

where the underlined term in the third line vanishes and in the last line the fact has been exploited that the integrand (including the inner integral) is an even function of x. Inserting the last line into equation (A.1) and simplifying the denominator by using the symmetry of the integrand, one arrives finally at equation (33).

Appendix B. Numerical evaluation of the quadratures

The numerator N of the expression for the diffusion coefficient has the structure

$$N = \int_{0}^{v_\mathrm{m}} \mathrm{d}x\, g_1(x) \left(\int_{x}^{v_\mathrm{m}} \mathrm{d}y\, g_2(y) \right)^2, \tag{B.1}$$

where the functions $g_1(x)$, $g_2(x)$ possibly diverge at the boundary $v = v_\mathrm{m}$. The integrals can be split in parts that still have to be done numerically and others that extend over small neighborhoods around $v = v_\mathrm{m}$:

$$N = \int_{0}^{v_\mathrm{m}-\varepsilon} \mathrm{d}x\, g_1(x) \left(\int_{x}^{v_\mathrm{m}-\varepsilon} \mathrm{d}y\, g_2(y) + \int_{v_\mathrm{m}-\varepsilon}^{v_\mathrm{m}} \mathrm{d}y\, g_2(y) \right)^2 \int_{v_\mathrm{m}-\varepsilon}^{v_\mathrm{m}} \left(\int_{x}^{v_\mathrm{m}} \mathrm{d}y\, g_2(y) \right)^2. \tag{B.2}$$

The denominator can be split in a similar fashion and the diffusion coefficient can be rewritten as

$$D_\mathrm{eff} = 2rA^2 \frac{I_1 + 2K_2 I_2 + K_2^2 I_3 + K_3}{K_4 + I_4}, \tag{B.3}$$

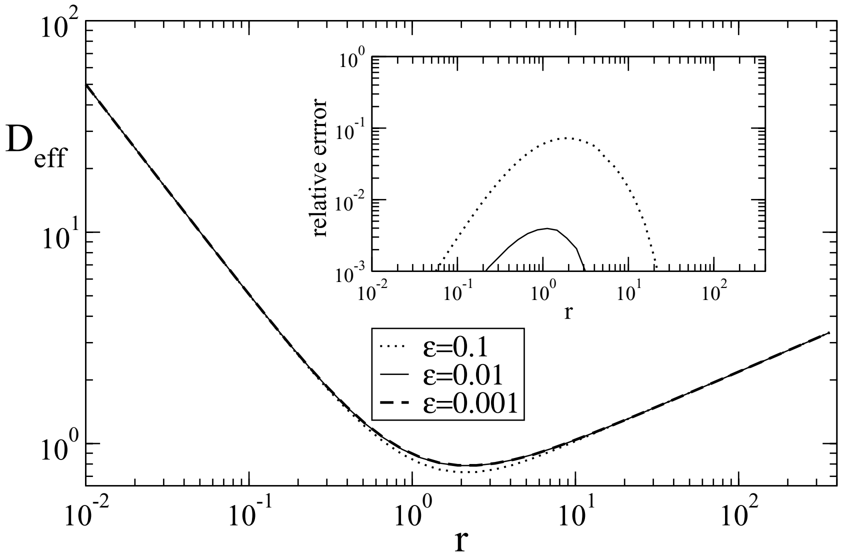

Figure B.1. Diffusion coefficient according to equation (B.3) for a quintic friction law $f(v) = -\gamma v^5$ versus rate with $\gamma = 1$, $A = 1$, and for different values of the numerical parameter ε as indicated. The curves for $\varepsilon = 10^{-2}$ and 10^{-3} agree within line thickness. The inset shows the relative deviation between the curves for $\varepsilon = 10^{-1}$ (dotted line) and $\varepsilon = 10^{-2}$ (solid line) to the curve for $\varepsilon = 10^{-3}$. For $\varepsilon = 10^{-2}$ this relative error is below 0.4%.

where

$$I_1 = \int_0^{v_\mathrm{m}-\varepsilon} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)} \left(\int_x^{v_\mathrm{m}-\varepsilon} \mathrm{d}y \frac{\mathrm{e}^{-U(y)}}{A^2 - f^2(y)} y \right)^2, \tag{B.4}$$

$$I_2 = \int_0^{v_\mathrm{m}-\varepsilon} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)} \int_x^{v_\mathrm{m}-\varepsilon} \mathrm{d}y \frac{\mathrm{e}^{-U(y)}}{A^2 - f^2(y)} y, \tag{B.5}$$

$$I_3 = \int_0^{v_\mathrm{m}-\varepsilon} \mathrm{d}x \frac{\mathrm{e}^{U(x)}}{A^2 - f^2(x)}, \tag{B.6}$$

$$I_4 = \int_0^{v_\mathrm{m}-\varepsilon} \mathrm{d}z \frac{\mathrm{e}^{-U(z)}}{A^2 - f^2(z)}. \tag{B.7}$$

For the analytical estimation of the terms K_i, an expression for the potential close to the boundary is required (similar techniques as employed in the following, have also been used to study the divergence of the stationary probability density, see e.g. [14, 20]). For $v_\mathrm{m} - \varepsilon < x < v_\mathrm{m}$ where ε is small, one can write the potential as follows:

$$U(x) = -2r \int_0^{v_\mathrm{m}-\varepsilon} \mathrm{d}y \frac{f(y)}{A^2 - f^2(y)} - 2r \int_{v_\mathrm{m}-\varepsilon}^{x} \mathrm{d}y \frac{f(y)}{A^2 - f^2(y)} \tag{B.8}$$

$$= U_\varepsilon - 2r \int_{v_\mathrm{m}-\varepsilon}^{x} \mathrm{d}y \frac{f(v_\mathrm{m}) + f'(v_\mathrm{m})(y - v_\mathrm{m}) + \cdots}{A^2 - [f^2(v_\mathrm{m}) + 2f(v_\mathrm{m})f'(v_\mathrm{m})(y - v_\mathrm{m}) + \cdots]}, \tag{B.9}$$

where I have abbreviated the first integral by U_ε and expanded the numerator and denominator in the second integral around $y = v_{\rm m}$. Taking only the leading orders and using the fact that $f(v_{\rm m}) = -A$ yields

$$\begin{aligned} U(x) &\approx U_\varepsilon - \frac{r}{f'(v_{\rm m})} \int_{v_{\rm m}-\varepsilon}^{x} {\rm d}y \frac{1}{(v_{\rm m} - y)} \\ &= U_\varepsilon + \frac{r}{f'(v_{\rm m})} \ln\left(\frac{v_{\rm m} - x}{\varepsilon}\right), \quad \text{for } v_{\rm m} - \varepsilon < x < v_{\rm m}, \end{aligned} \tag{B.10}$$

and for the exponential of the potential one obtains

$$e^{-U(x)} \approx e^{-U_\varepsilon} \left(\frac{v_{\rm m} - x}{\varepsilon}\right)^{\alpha}, \quad \alpha = -\frac{r}{f'(v_{\rm m})}. \tag{B.11}$$

Here α is a positive exponent because $f'(v_{\rm m}) < 0$. With this expression one can calculate an approximation for K_2 in equation (B.3):

$$\begin{aligned} K_2 &= \int_{v_{\rm m}-\varepsilon}^{v_{\rm m}} {\rm d}x\, e^{-U(x)} \frac{x}{A^2 - f^2(x)} \\ &\approx \int_{v_{\rm m}-\varepsilon}^{v_{\rm m}} {\rm d}x\, e^{-U_\varepsilon} \frac{x}{A^2 - [f^2(v_{\rm m}) + 2f(v_{\rm m})f'(v_{\rm m})(y - v_{\rm m}) + \cdots]} \left(\frac{v_{\rm m} - x}{\varepsilon}\right)^{\alpha} \\ &\approx \int_{v_{\rm m}-\varepsilon}^{v_{\rm m}} {\rm d}x\, e^{-U_\varepsilon} \frac{v_{\rm m}}{2f(v_{\rm m})f'(v_{\rm m})(v_{\rm m} - x)} \left(\frac{v_{\rm m} - x}{\varepsilon}\right)^{\alpha} \\ &= \frac{v_{\rm m}}{2Ar} e^{-U_\varepsilon} \end{aligned} \tag{B.12}$$

By similar expansions one finds

$$\begin{aligned} K_3 &= \int_{v_{\rm m}-\varepsilon}^{v_{\rm m}} {\rm d}x \frac{e^{U(x)}}{A^2 - f^2(x)} \left(\int_{x}^{v_{\rm m}} {\rm d}y\, e^{-U(y)} \frac{y}{A^2 - f^2(y)} \right)^2 \\ &\approx \frac{v_{\rm m}^2}{8A^3 r^3} e^{-U_\varepsilon}, \end{aligned} \tag{B.13}$$

$$\begin{aligned} K_4 &= \int_{v_{\rm m}-\varepsilon}^{v_{\rm m}} {\rm d}x \frac{e^{-U(x)}}{A^2 - f^2(x)} \\ &\approx \frac{1}{2Ar} e^{-U_\varepsilon}. \end{aligned} \tag{B.14}$$

For a finite ε, equation (B.3) is certainly an approximation that, however, yields for sufficiently small ε numerical results that do not change upon further reduction of ε (cf. figure B.1).

For small rate and fixed amplitude, the dominating term in equation (B.3) will be K_3 in the numerator and K_4 in the denominator. Neglecting the other terms in the fraction, the diffusion coefficient approaches for vanishing rate the simple expression equation (40) for a particle with a velocity following a dichotomous noise.

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