Three, more than two, at least three: What's the difference?

Manfred Krifka
Humboldt University Berlin
& ZAS Berlin
krifka@rz.hu-berlin.de

Scuola Normale Superiore, Pisa
March 3, 2006

By way of introduction...

What is the meaning of *three, at least three, more than two* in
- a. *John had three martinis.*
- b. *John had at least three martinis.*
- c. *John had more than two martinis.*

Answer of Generalized Quantifier (GQ) Theory (Barwise & Cooper 1982, etc.):
-- *three, at least three, more than two* are all quantificational determiners;
-- they have the same meaning, hence a./b./c. mean the same.

Analysis in GQ theory, where
⟦*martinis*⟧ = λx[x is a martini] = m
⟦*John had*⟧ = λx[John had x] = jh

a. ⟦*three*⟧ (⟦*martinis*⟧)(⟦*John had*⟧) = $|m \cap jh| \geq 3$
not $=3$ (*exactly three*), as continuation *if not four* is possible;
meaning component $\ngtr 3$ due to scalar implicature, not literal meaning.

b. ⟦*at least three*⟧ (⟦*martinis*⟧)(⟦*John had*⟧) = $|m \cap jh| \geq 3$

c. ⟦*more than two*⟧ (⟦*martinis*⟧)(⟦*John had*⟧) = $|m \cap jh| > 2$, i.e. ≥ 3

Result of this talk:
-- None of *three, at least three, more than two* are quantificational determiners;
-- they all have different meanings.

---

three as a numerical adjective

Analysis of *three* as a numeral (Verkuyl 1981, Link 1983):
$[_{NP}$*three* $[_{N}$*martinis*]]: $\lambda x[m(x) \wedge |x|=3]$
$[_{DP}\ ø_{\exists}\ [_{NP}$*three* $[_{N}$*martinis*]]]: $\lambda P \exists x[m(x) \wedge |x|=3 \wedge P(x)]$
$[_{S}$ *John* $[_{VP}$ *had* $[_{DP}\ ø_{\exists}\ [_{NP}$ *three martinis*]]]]: $\exists x[m(x) \wedge |x|=3 \wedge h(j, x)]$

Advantages:

- Allows for compositional analysis of $[_{DP}$ *the* $[_{NP}$ *three martinis*]]: $\sigma(\lambda x[m(x) \wedge |x|=3])$,
  sum of all entities in predicate, provided that it falls under the predicate,
  presupposition: there are exactly three martinis,
  reference to the single object in the extension of this predicate.
- Allows for existential analysis of bare nouns: $[_{DP}\ ø_{\exists}\ [_{NP}$ *martinis*]]
- Can be extended to a measure term analysis (cf. Krifka 1995)
  in which *martinis* denotes a measure function from entities to numbers:
  $m(x)=n$ iff x are n martinis;
  allows modelling cases like
  $[_{S}$ *John* $[_{VP}$ *had* $[_{DP}\ ø_{\exists}\ [_{NP}$ *two a half martinis*]]]]: $\exists x[m(x)=2.5 \wedge h(j, x)]$

Scalar implicature with *three*

In many contexts,
John had three martinis is understood as *John had exactly three martinis*.

Additional meaning component is a scalar implicature, as in GQ theory:
John had three martinis, if not four

Standard account of scalar implicature
(Grice 1967, Horn 1972, Gazdar 1979, Levinson 1984):
John had three martinis.
a. Meaning: $\exists x[m(x)=\underline{3} \wedge h(j,x)]$
b. Scalar implicature: $\neg\exists n>3 \exists x[m(x)=\underline{n} \wedge h(j,x)]$

Generation of scalar implicature:
Alternatives to *three*: *... two, three, four, ...*
Selection of strongest proposition by Maxim of Quantity:
Say as much as you can.
Exclusion of stronger alternatives by Maxim of Quality.
Don't say anything for which you lack evidence.

Question:
Why do *more than two* and *at least three* do not trigger this implicature?
A particularly challenging problem for the GQ account!

Why no scalar implicatures with *at least* / *more than*

Following Krifka (1999), ‘At least some determiners aren’t determiners’:

- Focus introduces background-focus partitioning, with alternatives (von Stechow 1981, 1990; Jacobs 1984): $\langle B, F, A\rangle$
  John had thrée$_F$ *martinis*: $\langle \lambda n \exists x\, [m(x)=n \wedge h(j,x)], 3, \{1, 2, 3, 4, ...\}\rangle$
- Illocutionary act is based on Background applied to Focus: $B(F)$, implicatures are denegations of illocutionary acts with respect to alternatives, e.g. if $X \in A$ and $X \neq F$, then $B(X)$ is not asserted (because it is too weak, or too strong, etc.)

  More specifically: ASSERT [*John had thrée*$_F$ *martinis*]:
  a. ASSERT $\exists x[m(x)=n \wedge h(j,x)]$
  b. Implicature: Every stronger alternative is not asserted:
  $\forall n>3$ ~ASSERT $\exists x[m=n \wedge h(j,x)]$,
- Determiners *at least / more than / at most / less than* associate with focus:
  e.g. (roughly): ⟦*more than*⟧ $(\langle \lambda n \lambda X[...n...], F, A\rangle) = \lambda X\ \exists n \in A, n>F\ [...n...]$
- No focus remains to be exploited at illocutionary level.
  ASSERT [*John had* [*more than*]$_1$ *twó*$_{F1}$ *martinis*]:
  ASSERT $\exists n>2 \exists x[m(x)=n \wedge h(j,x)]$

Under this analysis:

⟦*thrée*$_F$ *martinis*⟧ ≠ ⟦*at least thrée*$_F$ *martinis*⟧ , ⟦*more than twó*$_F$ *martinis*⟧ ,
but ⟦*at least thrée*$_F$ *martinis*⟧ = ⟦*more than twó*$_F$ *martinis*⟧ ,
at least if we only allow for natural number determiners, e.g. not *two and a half*

Geurts & Nouwen: ⟦*at least three*⟧ ≠ ⟦*more than two*⟧

Geurts & Nouwen (2005), ‘*At least* et al.: The semantics of scalar modifiers’
more than two, less than three: comparative modifiers
at least three, at most three: superlative modifiers

Meanings differ in subtle ways:

- Intuitive inference patterns:
  (a) *John had three martinis.*
  ⊨ *John had more than two martinis.*
  ⊭ *John had at least three martinis.* (lack of scalar implicature of (a))
  ⊨ *John had fewer than five martinis.*
  ⊭ *John had at most four martinis.* (contradicts scalar implicature of (a))
- Superlative modifiers have fewer distributional restrictions (cf. also Kay 1992):
  *John had three martinis at most / *fewer than.*
  *At least/*More than, John had three martinis.*
  *Mary danced with at most / *more than John.*
- Missing readings:
  You may have fewer than three martinis.
  a. ‘You are allowed to have fewer than three martinis (but you may have more)’
  b. ‘You may have up to two martinis (but not more)’

  You may have at most two martinis.
  only b. ‘You may have up to two martinis (but not more)’

Geurts and Nouwen’s Proposal:

Comparative modifiers:

Analysis as before; narrow-scope NP modifiers,
e.g. *John had* $[_{DP}\ ø_{\exists}\ [_{NP}$ *more than* $[_{NP}$ *thrée*$_F$ *martinis*$]]]$
$\exists n>3 \exists x[m(x)=n \wedge h(j,x)]$

Superlative modifiers:

Analysis as modal constructions, here adapted and simplified for our example

- *John had at least* $[_{DP}\ ø_{\exists}\ [_{NP}$ *thrée*$_F$ *martinis*$]]$
  $\Box\, \exists x[m(x)=3 \wedge h(j,x)]$
  $\wedge\ \exists n>3 \Diamond \exists x[m(x)=n \wedge h(j,x)]$
  ‘It is (epistimically) necessary that John had three martinis, and it is possible that he had more than three martinis.’
- *John had at most* $[_{DP}\ ø_{\exists}\ [_{NP}$ *thrée*$_F$ *martinis*$]]$
  $\Diamond \exists x[m(x)=3 \wedge h(j,x)]$
  $\wedge\ \neg\exists n>3 \Diamond \exists x[m(x)=n \wedge h(j,x)]$
  ‘It is possible that John had three martinis, but it is not possible that he had more than three martinis.’

Explanation of differences between comparative and superlative modifiers

- Intuitive inference patterns:
  John had three martinis.
  meaning: $\exists x[m(x)=3 \wedge h(j,x)]$
  implicature: $\neg\exists n>3 \exists x[m(x)=n \wedge h(j,x)]$

  ⊭ *John had at least three martinis.*
  $\Box[......] \wedge \exists n>3 \Diamond \exists x[m(x)=n \wedge h(j,x)]$
  inconsistent with implicature!

  ⊭ *John had at most four martinis.*
  $\Diamond \exists x[m(x)=4 \wedge h(j,x)] \wedge \neg\exists n>4 \Diamond[......]$
  inconsistent with implicature!
- Distributional restrictions
  At most, John had thrée$_F$ *martinis.*
  John had at most thrée$_F$ *martinis.*
  John had thrée$_F$ *martinis at most.*

  as a modal operator,
  at most has scope
  over the sentence.

  Mary danced with at least
  */ *more than John*

  modal operator does not require
  number scale;
  Kay 1993 for NP, AP, AdvP, VP uses
- Missing readings
  John may have at most two martinis.
  only possible reading: $\Diamond \boxdot \exists x[m(x)=3 \wedge h(j,x)] \wedge \neg\exists n>3 \Diamond \boxdot \exists x[m(x)=n \wedge h(j,x)]$,
  where $\Diamond$: epistemic possibility, $\boxdot$: deontic necessity,
  deontic operator cannot scope over epistemic operator

Problems of the analysis of Geurts & Nouwen

- The comparative morphology for comparative modifiers is captured by the analysis:
  e.g. *John had more than three martinis*
  $\exists x[m(x) \underline{> 3} \wedge h(j,x)]$
  But the superlative morphology for superlative modifiers is not captured by the analysis:
  e.g. *John had at least three martinis.*
  $\Box \exists x[m(x) = 3 \wedge h(j,x)] \wedge \exists n{>}3 \Diamond \exists x[m(x) = n \wedge h(j,x)]$
- There are restrictions on embeddings of superlative quantifiers that are difficult to explain under the modal analysis:
  Whenever you have less than 50 € in your pocket, go to the bank to get more.
  $^?$ *Whenever you have at most 50 € in your pocket, go to the bank to get more.*
- The analysis does not predict that *at most three* licenses NPIs, as no general downward-entailing context is created:
  At most three people have ever been in this cave.
  * It is possible that 3 people have ever been in this cave ∧
  o.k.: ∃n>3 ¬It is possible that n people have ever been in this cave

A new analysis:
at most / *at least* as operators on the illocutionary level

Basic idea for superlative modifiers:

- Upper-bound superlatives:
  John had at most thrée$_F$ *martinis.*
  'The highest n such that ∃x[m(x)=n ∧ h(j,x)] can be asserted is n = 3',
  that is, *John had n martinis* cannot be asserted for n > 3;
  intuitive reason: this would definitely be too strong,
  i.e. it would most likely be false.
- Lower-bound superlatives:
  John had at least thrée$_F$ *martinis.*
  'The lowest n such that ∃x[m(x)=n ∧ h(j,x)] can be asserted is n = 3'
  that is, *John had n martinis* cannot be asserted for n < 3;
  intuitive reason: this would definitely be too weak,
  i.e. it would most likely give raise to the wrong implicature
  that drank only less than three martinis.

Relation Illocutionary Strength – Superlative Operators

Example situation:
Assume there is **very good evidence** that your guest John had **two** martinis
(you have seen it with your own eyes),
there is **good evidence** that John had **three** martinis
(someone told you),
and there is **weak evidence** that John had **four** martinis
(a quantity of four martinis is missing from your bar.

In this situation, one could say:

John had at least two martinis.	(≃ strong assertion)
John had three martinis.	(≃ neutral assertion)
John had at most four martinis.	(≃ weak assertion)

Cf. notion of illocutionary strength
by Searle & Vanderveken 1985, Vanderveken 1990

How can we explain this relation of superlative operator to illocutionary strength?

Neutral, weak and strong assertions

The simple view: H.P. Grice, Maxim of Quality:
"Contribute only what you know to be true.
Do not contribute false things.
Do not say things for which you lack evidence."

But: We also often say things that we consider just likely,
or that we know of potentially unreliable sources.

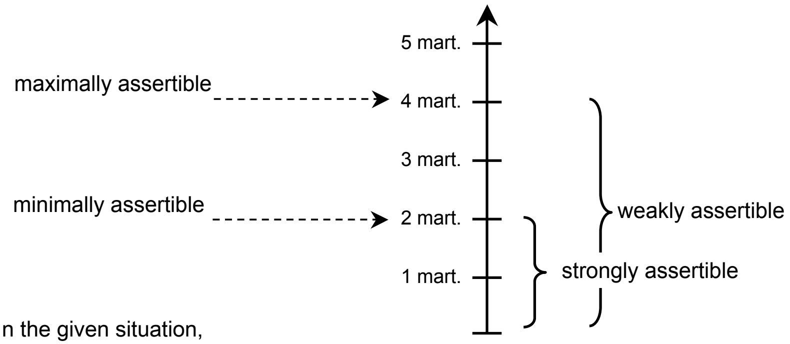
Proposal: Distinguish between neutral, strong and weak assertions.

- Neutral assertion:
  ASSERT(N) 'John had martinis'
  Maxim of Quality holds, Speaker has good evidence, is liable for the claim.
- Weak assertion:
  ASSERT(W) 'John had martinis'
  Maxim of Quality holds in a more relaxed way,
  Speaker has weaker evidence and is not fully liable for the claim.
- Strong assertion:
  ASSERT(S) 'John had martinis'
  Maxim of Quality holds in a strict way, Speaker fully liable fo the claim.

Strength may be indicated, e.g. by *presumably*, *certainly*

Cf. also Potts (2005): Rated Quality to improve Relevance

Weak and strong assertions and superlative operators



In the given situation,

- the maximal value that is assertible is *four martinis*,
  hence: *John had at most four martinis*
  This asserts only <u>weakly</u> that John had four martinis (the evidence is weak)
- the minimal value that is assertible, is *two martinis,*
  factoring in a maximizing scalar implicature,
  hence: *John had at least two martinis.*
  This asserts <u>strongly</u> that John had two martinis (the evidence is strong)

A closer look at simple scalar implicatures: *thrée*

Neutral assertion with scalar implicature:
ASSERT(N)$_1$ *John had thrée$_{F1}$ martinis*
a. ASSERT(N) $\exists x[m(x)=3 \wedge h(j,x)]$
b. & $\forall n>3$ ~ASSERT(N) $\exists x[m(x) =n \wedge h(j,x)]$

in short:
SC-ASSERT(N) $\exists x[m(x)=\underline{3} \wedge h(j,x)]$
It is neutrally asserted that John had three martinis,
and neutrally implicated that he didn't have more martinis.

where
& is illocutionary conjunction (cf. Krifka 2001)
~ is illocutionary denegation (cf. Searle 1969, Vanderveken 1990),
speaker refrains from performing the illocutionary act.

Notice:
Illocutionary act and denegations of illocutionary acts are of the same strength;
in our example,
the implicature does not deny that there may be *weak* evidence
that John had more than three martinis.

A closer look at lower-bound superlative operators

Basic idea: *John had at least$_1$ twó$_{F1}$ martinis.*
'The lowest n such that $\exists x[m(x)=n \wedge h(j,x)]$ can be asserted is n = 2'

Proposal: *at least* expresses scalar assertion of focus + denegation of lower values.
a. SC-ASSERT(•) $\exists x[m(x)=\underline{2} \wedge h(j,x)]$
b. & $\forall n<2$ ~[SC-ASSERT(•) $\exists x[m(x)=\underline{n} \wedge h(j,x)]]$
where • stands for N, W, S

a. – Speaker asserts with a certain strength that John had two martinis,
– and implicates with that strength that he did not have more than two martinis.

b. For numbers n<2 speaker explicitly does not:
– assert with a certain strength that John had n martinis
– and implicate with that strength that John had not more than n martinis.

The only coherent way to specify • in (a) is as S,
a. SC-SSERT(S) $\exists x[m(x)=\underline{2} \wedge h(j,x)]$
b. & $\forall n<2$ ~[SC-ASSERT(W∨N∨S) $\exists x[m(x)=\underline{n} \wedge h(j,x)]]$
'It is <u>strongly</u> asserted that John had 2 martinis (implicated: not more),
and not asserted with any strength that John had 1 martini (implicated: not more)'
Reason for non-assertion: resulting implicature would be false, given the assertion.

Assume to contrary: Strength in (a) specified as W
'It is <u>weakly</u> asserted that John had 2 martinis (implicated: not more),
and not asserted with any strength that John had 1 martini (implicated: not more)'
This implicature would not be false for strong assertion,
as there might in fact be strong evidence only for one martini.
Hence, *at least* cannot be interpreted as weak (or neutral) assertion.

A closer look at upper-bound superlative operators

Basic idea: *John had at most$_1$ fóur$_{F1}$ martinis.*
'The highest n such that $\exists x[m(x)=n \wedge h(j,x)]$ can be asserted is n = 4'

Proposal: *at most* expresses scalar assertion of focus + denegation of higher values.
a. SC-ASSERT(•) $\exists x[m(x)=\underline{4} \wedge h(j,x)]$
b. & $\forall n>4$ ~ SC-ASSERT(•) $\exists x[m(x)=\underline{n} \wedge h(j,x)]$
where • stands for an evidential qualification of assertion like N, W, S

a. – Speaker asserts with a certain strength that John had four martinis,
– and implicates with that strength that he did not have more than four martinis.

b. For numbers n>4 speaker explicitly does not:
– assert with a certain strength that John had n martinis
– and implicate with that strength that John had more than n martinis.

The only coherent way to specify • in (a) is as W,
a. SC-ASSERT(W) $\exists x[m(x)=\underline{4} \wedge h(j,x)]$
b. & $\forall n>4$ ~ SC-ASSERT(•) $\exists x[m(x)=\underline{n} \wedge h(j,x)]]$

'It is <u>weakly</u> asserted that John had 4 martinis (implicated: not more),
and not asserted with any strength that John had >4 martini (implicated: not more)'
Reason for non-assertion: its implicature would be false, given the assertion.

Specifying • in (a) is as S is not possible,
as then assertions based on higher numbers would be possible as weak assertions.

Explanation of observations of G&N

- Inference patterns:
  John had thrée$_F$ martinis. SC-ASSERT(N) $\exists x[m(x)=\underline{3} \wedge h(j,x)]$
  $\nvDash$ *John had at least thrée$_F$ martinis.* SC-ASSERT(S) $\exists x[m(x)=\underline{3} \wedge h(j,x)]$ & $\forall n>3$ ~SC-ASSERT(S) $\exists x[m(x)=\underline{n} \wedge h(j,x)]$
  Something that is normally assertible might not be strongly assertible.
  $\nvDash$ *John had at most thrée$_F$ martinis.* SC-ASSERT(W) $\exists x[m(x)=\underline{3} \wedge h(j,x)]$ & $\forall n>3$ ~SC-ASSERT(W) $\exists x[m(x)=\underline{n} \wedge h(j,x)]$
  Something normally assertible might not be the upper bound that is weakly assertible.
- Distributional restrictions
  At most, John had thrée$_F$ martinis.
  John had at most thrée$_F$ martinis.
  John had thrée$_F$ martinis at most.
  superlative modifiers affect illocutionary level hence have sentential scope
- Missing readings
  You may drink at most two martinis.
  Similar reasoning: superlative modifers affect illocutionary level, deontic operator is part of proposition (but see later for commands, permissions)

Explanation of observations that G&N could not explain:
Superlative morphology of *at least / at most*

Natural explanation:
John had at most three martinis
= The most speaker is willing to assert
out of the alternatives *John had ...2/3/4... martinis*
is: *John had 3 martinis*
John had at least three martinis
= The least speaker is willing to asser
out of the alternatives *John had ...2/3/4... martinis*
is: *John had 3 martinis*

Explanation of observations that G&N could not explain:
Licensing of NPIs by *at most*

Example: *At most three people have ever been in this cave (in the last century)*

Explanation of NPI distribution, Krifka (1995):

- NPIs introduce alternatives and denote the smallest or most general alternative.
  e.g *ever ... in the last century* :
  Alternatives: more or less specific times, $\{t \mid t \subseteq \text{last\_century}\}$
  Meaning: The least specific time, last_century
- Alternatives are exploited at the level of illocutionary operators (cf. scalar implicature):
  ASSERT [*no person has ever been in this cave in the last century*]
  ASSERT $[\neg\exists x[\text{person}(x) \wedge \text{in\_cave}(x, \text{last\_century})]$ &
  $\forall t \subseteq \text{last\_century}$ [~ ASSERT $[\neg\exists x[\text{person}(x) \wedge \text{in\_cave}(x, t)]]$
  Reason for non-assertion:
  Alternatives are less informative than the actual assertion:
  $\neg\exists x[\text{person}(x) \wedge \text{in\_cave}(x, \text{last\_century})] \Rightarrow$
  $\forall t \subseteq \text{last\_century}$ [~ ASSERT $[\neg\exists x[\text{person}(x) \wedge \text{in\_cave}(x, t)]]$
  This holds for downward-entailing contexts in general.
- In general:
  NPI indicates that the “strongest” claim is made among the alternatives.

Explanation of observations that G&N could not explain:
Licensing of NPIs by *at most*

NPIs o.k. with upper-bound operators,
as speaker indicates that the assertion is the strongest proposition
that he could still reasonably defensible

Worked-out example:
At most three people have ever been in this cave (in the last century)
ASSERT(W) $\exists x[\text{person}(x)=3 \wedge \text{in\_cave}(x, \text{last\_cent})]$
& $\forall n>3$ ~ ASSERT($\cdot$) $\exists x[\text{person}(x)=n \wedge \text{in\_cave}(x, \text{last\_cent})]$
& $\forall t \subset \text{last.cent}$ ~ [ASSERT(W) $\exists x[\text{person}(x)=3 \wedge \text{in\_cave}(x, t)]$
& $\forall n>3$ ~ ASSERT($\cdot$) $\exists x[\text{person}(x)=n \wedge \text{in\_cave}(x, t)]$

It is asserted that at most n people have been in this cave in the last century;
for times $t \subset$ last.cent: it is not asserted
that at most n people have been in this cave in the last century
(because these assertions would be weaker than the assertion actually made)

This is possible only as a weak assertion;
speaker indicates:
Only when lowering standards of verifyability as much as possible
can one defend the claim that three people have been in this cave.

It follows from the logic of weak and stong assertions that
* *At least three people have ever been in this cave (in the last century),*
as lower-bound operators indicate indicate strong assertions
and indicate that more general weak assertions would be possible.

Explanation of observations that G&N could not explain: Embeddability of comparative vs. superlative operators

Examples:

- *If you have less than 50 € in your pocket,*
  go to the bank to get more.
- $^?$ *If you have at most 50 € in your pocket, go to the bank to get more.*
  o.k. as quotation
- *If you ever have more than one thousand Euros in your pocket,*
  bring it to the bank.
- $^?$*If you ever have at least one thousand Euros in your pocket,*
  bring it to the bank.

Under an analysis of superlative operators as operators on the illocutionary level,
one should expect that they cannot be embedded within the propositional level.

There are differences between comparative and superlative operators
(cf. example above, also discussion in G&N),
but embeddings of superlative operators are often not as bad as predicted.

There is independent evidence that illocutionary operators can be embedded,
cf. Krifka (1995), Chierchia (2001) on scalar implicatures in embedded sentences:
If you have 50 € in your pocket, you should give me 25 and keep 25 for yourself.
(= *If you have 50 € (and not more) in your pocket, ...*)

Other illocutionary operators: Commands

Commands:
Speaker wants the hearer to do something against hearer's interest,
restricts option space of hearer.

Minimal and maximal commands:

(a) *Give me at least twó*$_F$ *euros.*
'The smallest n such that I ask you to give me n euros is n = 2'
'I ask you to give me 2 euros,
and I do not ask you to give me less than 2 euros (this would be too little)'
'I am minimally (sufficiently) satisfied if you give me 2 euros,
more satisfied if you give me more.'

(b) $^?$*Give me at most fóur*$_F$ *euros.*
'The greatest n such that I ask you to give me n euros is n = 4'
'I ask you to give me 4 euros,
and I do not ask you to give me more than 4 euros (this would be too much)'
'I am maximally satisfied if you give me 4 euros.'

(b) is less natural, most likely because maximal satisfaction is a strange concept,
o.k. in politeness contexts with distorted interests.

Minimal commands analyzed

Scalar implicature in commands:
Give me twó$_F$ *euros.*
COMMAND(•) g(y,m,2€) & ∀n>2 ~COMMAND(•) g(y,m,n€),
'I ask you to give me 2 euros, and I do not ask you to give me more'
In short: COMMAND(•)$_i$ g(y,m,$\underline{2}$€)

Minimal commands:
Give me at least twó$_F$ *euros.*
COMMAND(•) g(y,m,$\underline{2}$€) & ∀n<2 ~COMMAND(•) g(y,m,$\underline{n}$€)
'I ask you to give me 2 euros (and not more),
but I do not ask you to give me 1 euro (and not more)'
Reason: Implicature of second clause would contradict command of first.

Consistent specification of strength:
COMMAND($\underline{W}$) g(y,m,$\underline{2}$€) & ∀n<2 ~COMMAND($\underline{S \vee W}$) g(y,m,$\underline{n}$€)
'The minimum that would satisfy me is: you give me 2 euros,
less than 2 would not satisfy me, minimally or maximally.'

* COMMAND($\underline{S}$) g(y,m,$\underline{2}$€) & ∀n<2 ~COMMAND($\underline{S \vee W}$) g(y,m,$\underline{n}$€)
'The maximum that would satisfy me is: you give me 2 euors,
less than 2 would not satisfy me maximally or minimally'
-- coherent only in case: max=min.

Hence: *at least* specifies minimal commands, they express the least that I want;
at most problematic in commands because I'm certainly not against getting more.

Other illocutionary operators: Permissions

Permissions:
Speaker wants to allow the hearer to do something in his interest,
Speaker increases option space of hearer.

Maximal and minimal permissions:

(a) *You may take at most fóur*$_F$ *euros.*
'The greatest n such that I allow you to take n euros is n = 4'
'I allow you to take 4 euros,
and I do not allow you to take more than 4 euros (this would be too much)'
'I satisfy you most (within given limits) if I allow you to take 4 euros,
I satisfy you less if I allow you to take less.'

(b) $^?$*You may take at least twó*$_F$ *euros.*
'The smallest n such that I allow you to take n euros is n = 2'
'I allow you to take 2 euros,
and I do not allow you to take less than 2 euros (this would be too little)'
'I satisfy you least (within given limits) if I allow you to take 2 euros,
I satisfy you less if I allow you to take less.'

(b) is less natural because permissions normally are downward-entailing:
If I allow you to take n euros, you can also take m euros, m<n.
o.k. in a description of a previously uttered permission

Maximal permissions analyzed

Scalar implicature in permissions:
You may take fóur$_F$ *euros.*
$\text{PERM}(\cdot)\ t(y,4\text{€})\ \&\ \forall n{>}2\ {\sim}\text{PERM}(\cdot)\ t(y,n\text{€})$,
‘I allow you to take 4 euros, and I do not allow you to take more’
In short: $\text{PERM}(\cdot)\ t(y,\underline{4}\text{€})$

Maximal permissions:
You may take at most fóur$_F$ *euros. / up to fóur*$_F$ *euros.*
$\text{PERM}(\cdot)\ t(y,\underline{4}\text{€})\ \&\ \forall n{>}2\ {\sim}\text{PERM}(\cdot)\ t(y,\underline{n}\text{€})$
‘I allow you to take 4 euros (and not more),
but I do not allow you to take >n euro (and not more)’

Consistent specification of strength:
$\text{PERM}(\underline{S})\ t(y,\underline{4}\text{€})\ \&\ \forall n{>}4\ {\sim}\text{PERM}(\underline{S{\vee}W})\ t(y,\underline{n}\text{€})$
‘The maximum that I allow you to do is: you take 4 euros,
I do not allow you to take more than 4, minimally or maximally.’
* $\text{PERM}(\underline{W})\ t(y,\underline{4}\text{€})\ \&\ \forall n{>}4\ {\sim}\text{PERM}(\underline{S{\vee}W})_i\ t(y,\underline{n}\text{€})$
‘The minimum that I allow you to do is: you take 4 euros,
I do not allow you to take more than 4, minimally or maximally’
-- coherent only in case: max=min.

Hence: *at most* specifies maximal permissions.

Speech Acts vs. Resultant Modalities

You must give me at least twó$_F$ *euros.*
Readings, cf. also Geurts & Nouwen:
a. Command: ‘Give me at least twó$_F$ euros.’
b. Description of obligation: ‘The minimal n such that you must give me n euros is 2’

Reading (a) (called “modal concord reading” in Geurts & Nouwen)
is derived in the same way as *Give me at least twó*$_F$ *euros.*

Equivalent reading is unavailable for *You must give me at most fóur*$_F$ *euros*
for the same reason as $^?$*Give me at most fóur*$_F$ *euros.*

Reading (b) (called “compositional reading” in Geurts & Nouwen, $\boxdot$: deontic necessity)
$\text{ASSERT}(\cdot)\ \boxdot\ g(y,m,\underline{2}\text{€})\ \&\ \forall n{<}2\ {\sim}\text{ASSERT}\ (\cdot)\ g(y,m,\underline{n}\text{€})$
‘It is asserted that you are required to give me two euros
(and not required to give me more)
and it is not asserted that you are required to give me one euro
(and not required to give me more)’

The only coherent specification, as before:
$\text{ASSERT}(\underline{W})\ \boxdot\ g(y,m,\underline{2}\text{€})\ \&\ \forall n{<}2\ {\sim}\text{ASSERT}\ (\underline{W{\vee}S})_i\ \boxdot\ g(y,m,\underline{2}\text{€})$

An equivalent reading is available for *You must give me at most fóur*$_F$ *euros*:
‘It is asserted that you are required to give me four euros,
and it is not asserted that you are required to give me more.’

Analysis of “modal concord reading”

Geurts & Nouwen:
Absorption of modal operators expressed by *must*
with first modal operator of *at least* into one, if compatible:
You <u>must</u> give me <u>at least</u> twó$_F$ *euros.*
$\underline{\Box\ \Box}\ [\ldots\ 2\text{€}\ \ldots]\ \&\ \Diamond\exists n{>}2\ [\ldots\ n\text{€}\ \ldots]$

Current analysis:
You must give me at least twó$_F$ *euros*,
asserts that an commitment state of the addressee exists that comes about
after the command *Give me at least two euros!*
hence it can be used as an indirect speech act with this meaning.

Similar analysis for:
I need at least twó$_F$ *euros.*
weak needs/strong needs

Wrapping things up...

Generalized Quantifier Theory:
Same meaning for *three, at least three, more than two*

We have seen that this is not the case:

- *three* is a number word specifying number argument of measure function, existential quantifier is independent of that.
- *more than two, fewer than four* are comparative modifications of this number argument
- *at least three, at most four* are superlative operators operating on a higher level, arguably on the level of illocutionary operators, expressing minimal and maximal illocutionary acts out of a set of alternative illocutionary acts.

Morale:
Natural Language “Quantification” is much richer, and very different
from what Generalized Quantifier Theory has suggested.

There is much more research to be done,
even for well-known languages like English, German, Italian.