On the semantics of lower and upper bounds

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Sinn Und Bedeutung IX

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1 Goals

Comparing two frequent means of modifying numerals:

▷ Comparative modifiers (CMs): *less than, more than*

▷ Superlative modifiers (SMs): *at most, at least*

Claims:

▷ CMs and SMs differ fundamentally in their semantic function

▷ CMs and SMs are not interchangeable: e.g. *at least four* ≠ *more than three*, *at most four* ≠ *less than five*

▷ CMs have a relatively simple semantics involving the selection of the upper (lower) half of a scale with respect to the denotation of the complement NP.

▷ SMs compare alternative propositions and have a modal and exhaustive semantics

2 Background

It is a common assumption that CMs and SMs have a similar function.

(1) a. *At least four students passed the exam.*
b. *More than three students passed the exam.*
c. "The number of students that passed the exam exceeds three"

(2) a. *At most four students passed the exam.*
b. *Less than five students passed the exam.*
c. "The number of students that passed the exam does not exceed four"

▷ *more than* and *at least* give a lower bound on the cardinality of the intersection of restrictor and scope

▷ *at most* and *less than* give an upper bound on the cardinality of the intersection of restrictor and scope

(3) a. $[\![\textit{at most } n]\!] = \lambda PQ.|P \cap Q| \leq n$
b. $[\![\textit{less than } n]\!] = \lambda PQ.|P \cap Q| < n$
c. $[\![\textit{at least } n]\!] = \lambda PQ.|P \cap Q| \geq n$
d. $[\![\textit{more than } n]\!] = \lambda PQ.|P \cap Q| > n$

2.1 Against the complex determiner view

Krifka 1999 : *at most* n*, less than* n, etc. are no complex determiners.

The relation expressed by the modifier does not necessarily apply to the number word.

(4) Q: *How many boys left?* (Krifka 1999)
A: *At least* [*three*]$_f$ *boys left.*

(5) Q: *Who left?* (Krifka 1999)
A: *At least* [*three boys*]$_f$ *left.*

Non-numeral scales:

(6) *Mary is at least an* [*associate*]$_f$ *professor.* (Krifka 1999)

(7) a. *I'm more than happy, I'm satisfied.*
b. *I'll be more than happy to drive you to the station.*

(8) *What!? John gave Mary six chocolates!? He should have given her at least six* [*flowers*]$_f$*!*
(the speaker thinks that flowers are the least kind of gift John should have given Mary six of)

2.2 CMs and SMs select sets of alternatives (Krifka 1999)

1. Focus triggers alternatives
2. CMs and SMs trigger an ordering relation among alternatives

The complement NP of the modifier has an alternative semantic value ($[\![\cdot]\!]^A$) expressing an ordering of alternatives to the normal semantic value of the NP.

(9) a. $[\![\textit{at least } \alpha\,]\!] = \cup\{X|\langle\, [\![\alpha, X]\!]\,\rangle \in [\![\alpha]\!]^A\}$
b. $[\![\textit{more than } \alpha]\!] = \cup\{X|\langle\, [\![\alpha]\!], X\rangle \in [\![\alpha]\!]^A \wedge X \neq [\![\alpha]\!]\,\}$

(10) $\overbrace{\overbrace{|\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-}^{\text{less than}}\alpha\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!-\!->}^{\text{at most}}$

(11) $|-----\alpha\underbrace{\underbrace{-----}_{\text{more than}}>}_{\text{at least}}$

Krifka 1999 does not distinguish between the semantic function of CMs and SMs. One of the predictions a theory like this makes is again that [*at least* [n N]] and [*more than* [$n-1$ N]] are interchangeable.

3 Focus sensitivity

Focus outside the complement NP:

(12) *John should give more than six flowers to* [*Mary*]$_f$.
- a. John should give seven or more flowers to Mary and not to someone else.
- b. #John should not give (just) six flowers to Mary. He should give six flowers to someone who is somehow 'more' than Mary.

(13) *John should give at least six flowers to* [*Mary*]$_f$.
- a. John should give six or more flowers to Mary and not to someone else.
- b. Mary is the least person who should be given six flowers by John.

(14) Q: *Who should John give six flowers to?*
A: *He shouldn't give them to Mary. #He should give more than six flowers to* [*Mary*]$_f$.

(15) Q: *Who should John give six flowers to?*
A: John should give at least six flowers to [Mary]$_f$.

(16) *(At least) John should (at least) give (at least) six flowers (at least) to Mary.*

(17)
- a. *John is at most of the opinion that* [*Bill*]$_f$ *should get a ticket for speeding.*
- b. *John is at most of the opinion that Bill should get* [*a ticket*]$_f$ *for speeding.*
- c. *John is at most of the opinion that Bill should get a ticket for* [*speeding*]$_f$.

Conclusion: Whereas CMs interact with focus in their complement NP only, SMs may interact with focus anywhere in the sentence.

3.1 A first proposal

3.1.1 Assumption

Focus triggers an ordering of a set of alternative semantic values for the focused constituent. I will systematically refer to the set of alternatives as A and refer to the ordering using one of $<_A, \leq_A, \geq_A, >_A$.

Furthermore, focus triggers a set of alternative propositions A_t, derived from A. This set is ordered similarly to A. The relation symbols $<_{A_t}, \leq_{A_t}, \geq_{A_t}, >_{A_t}$ refer to the ordering.

(18) a. *At least* [*three*]$_f$ *boys left.*
b. The set of alternatives consists of groups of boys.
c. $A = \{\langle \lambda x.\text{boy}(x) \wedge \#x = n, \lambda x.\text{boy}(x) \wedge \#x = m\rangle \mid n < m\}$

(19) a. *At least* [*three boys*]$_f$ *left.*
b. The set of alternatives consists of groups (including non-boy-groups).
c. $A =$ any salient and relevant ordering of groups

3.1.2 Comparative modifiers (along the lines of Krifka 1999)

Cf. (Krifka 1999) for compositional proposal.

(20) $lessthan \;=\; \lambda R.\lambda S.\; S \cap A <_A R$

(21) $morethan \;=\; \lambda R.\lambda S.\; S \cap A >_A R$

Example.

(22) a. *More than* [*three boys*]$_f$ *left.*
b. $\lambda x.\text{left}(x) >_A \lambda x.\text{boy}(x) \wedge \# = 3$

(23) a. *John gave more than three flowers to* [*Mary*]$_f$.
b. $\lambda x.\text{gave}(\text{j,m}, x) >_A \lambda x.\text{flower}(x) \wedge \#x = 3$

3.2 Superlative Modifiers

(24) $atmost\; p \;=\; \neg\exists p' >_{A_t} p : p'(w_0)$

(25) $atleast\; p \;=\; \neg\exists p' <_{A_t} p : p'(w_0)$

(26) a. *At least* [*three boys*]$_f$ *left.*
b. It is not the case that two boys left and it is not the case that one boy left and it is not the case that no boy left.

(27) a. *John gave at least four flowers to* [*Mary*]$_f$.
b. It is not the case that John gave four flowers to Sue, it is not the case that John gave four flowers to Ann, etc.

4 SMs and uncertainty

4.1 Belief reports containing SMs and CMs

(28) *John believes there are exactly nine VIPs at the party.*

a. $\Rightarrow$ *John believes there are more than six VIPs at the party.*

b. $\not\Rightarrow$ *John believes there are at least seven VIPs at the party.*

4.2 (Un)certainty

Context: There exists a bag which contains exactly six marbles. There are two children discussing the bag: Dicky and Billy.

Context C1	**Context C2**
Dicky knows about the bag's content.	Dicky doesn't know the bag's contents, but has seen someone put three marbles in the bag.

(29) Billy: *There are exactly two marbles in the bag.*

(30) a. Dicky: *That's not true. There are more than two marbles in the bag.* C1/C2

b. Dicky: *That's not true. There are at least three marbles in the bag.* #C1/C2

A similar observation has been made by Bart Geurts (pc, meeting Semantiknetzwerk 30/10/04):

(31) a. George had three beers, and therefore he had less than four beers.

b. George had three beers, and therefore he had at most three/four beers.

4.3 Intrinsic modality of superlative modifiers

(32) Bill pointing at John's son: *You can tell your son is an only child.*

a. John: *But he's not. I have more than one child.*

b. John: *#But he's not. I have at least two children.*

(33) a. *I have more than one child*: the world is such that I have more than one child.

b. *I have at least two children*: it is possible that I have two children, it is possible that I have three children, it is possible that I have four…

4.4 The story so far

▷ The naive view that CMs and SMs express upper (lower) bounds on the cardinality of a set is contradicted by the existence of certain focus effects

▷ The view that CMs and SMs share their basic semantics is contradicted by the fact that, unlike CMs, SMs consider propositional alternatives

▷ SMs appear to have a far richer semantics than CMs: they appear to be intrinsically modal.

5 Expressing lower (upper) bounds w.r.t. a modal

(34) *This lift can carry less than seven people.*

a. A load of six or less people will not cause the lift to crash.

b. 6 people is the upper bound on the capacity of this lift (i.e. a load containing 7 or more people will cause the lift to crash)

(35) *This lift can carry at most six people.*

a. #A load of six or less people will not cause the lift to crash.

b. 6 people is the upper bound on the capacity of this lift (i.e. a load containing 7 or more people will cause the lift to crash)

Similar examples.

(36) a. *The company may lose at most 400 million euros.* upper bound on loss

b. *The company may lose less than 400 million euros.* no upper bound on loss

Context: A discussion of the options a certain game gives a player in his/her first turn.

Rule Book RB1: In your first turn pick any number of card not exceeding 6	**Rule Book RB2**: In your first turn pick any number of cards you wish to pick

(37) a. *The game allows you to pick six cards in your first turn.* RB1/RB2

b. *The game allows you to pick less than seven cards in your first turn.* RB1/RB2

c. *The game allows you to pick at most six cards in your first turn.* RB1/#RB2

Conclusion: Whereas CMs have a weak, but possibly strengthened, interpretation with respect to a modal verb, SMs only have a strong interpretation.

6 Proposal

(38) $atleast_{\text{mod}}(p) \;=\; \text{mod}(\cup\{p' \mid p' \geq_{A_t} p\}) \;\wedge\; \neg\text{mod}(\cup\{p' \mid p' <_{A_t} p\})$

(39) $atmost_{\text{mod}}(p) \;=\; \text{mod}(\cup\{p' \mid p' \leq_{A_t} p\}) \;\wedge\; \neg\text{mod}(\cup\{p' \mid p' >_{A_t} p\})$

Prediction:

(40) a. $atmost_{\Box}$ { weak statement about upper half of scale, strong statement about lower half of scale

b. $atleast_{\Box}$ { strong statement about upper half of scale, weak statement about lower half of scale

c. $atmost_{\Diamond}$ { weak statement about lower half of scale, strong statement about upper half of scale

d. $atleast_{\Diamond}$ { strong statement about lower half of scale, weak statement about lower half of scale

6.1 Examples

(41) a. *This lift is carrying* [*six*]$_f$ *people*
b. The number of people this lift is carrying is (exactly) six (abbrev. carry6)

(42) a. *This lift can carry at most* [*six*]$_f$ *people.*
b. $\Diamond(\text{carry0} \vee \ldots \vee \text{carry6}) \wedge \neg\Diamond(\text{carry7} \vee \text{carry8} \ldots)$

(43) a. *This crane should be operated by at most* [*two*]$_f$ *people.*
b. $\Box(\text{0 operators} \vee \text{1 operator} \vee \text{2 operators}) \wedge \neg\Box(\text{3 operators} \ldots \text{4 operators} \ldots)$

(44) a. *This crane must be operated by at least* [*two*]$_f$ *people.*
b. $\Box(\text{2 operators} \vee \text{3 operators} \vee \ldots) \wedge \neg\Box(\text{0operators} \vee \text{1operator})$

Special case:

(45) a. *This lift can carry at least* [*six*]$_f$ *people.*
b. $\Diamond(\text{carry6} \vee \text{carry7} \vee \ldots) \wedge \neg\Diamond(\text{carry5} \vee \ldots \vee \text{carry0})$
c. $\Diamond(\text{max.capacity=6} \vee \text{max.capacity=7} \vee \ldots) \wedge$
$\neg\Diamond(\text{max.capacity=5} \vee \ldots \vee \text{max.capacity=0})$ least upper bound reading

6.2 Other examples

(46) a. Context: Dicky knows there are 8 marbles in the bag.
b. Dicky: *There are at least five marbles in the bag*.
c. $\Diamond$(five marbles $\vee$ six marbles $\vee \ldots$) $\wedge$
$\neg\Diamond$(four marbles $\vee$ three marbles $\vee \ldots \vee$ no diamond)

6.3 Non-natural scales

Scale of gifts:
a bottle of wine $<_A$ a box of chocolates $<_A$ a bunch of roses $<_A$ a pair of opera tickets $<_A$ jewelry

(47) a. *John may bring at most* [*a bunch of roses*]$_f$.
b. $\Diamond$(John brings roses $\vee$ John brings chocolates $\vee$ John brings wine) $\wedge$
$\neg\Diamond$(John brings tickets $\vee$ John brings jewelry)

(48) a. *John should bring at most* [*a bunch of roses*]$_f$.
b. $\Box$(John brings roses $\vee$ John brings chocolates $\vee$ John brings wine) $\wedge$
$\neg\Box$(John brings tickets $\vee$ John brings jewelry)

(49) a. *John may bring at least* [*a bunch of roses*]$_f$.
b. $\Diamond$(John brings roses $\vee$ John brings tickets $\vee$ John brings jewelry) $\wedge$
$\neg\Diamond$(John brings chocolates $\vee$ John brings wine)

(50) a. *John must bring at least* [*a bunch of roses*]$_f$.
b. $\Box$(John brings roses $\vee$ John brings tickets $\vee$ John brings jewelry) $\wedge$
$\neg\Box$(John brings chocolates $\vee$ John brings wine)

Potential problem: (49-b) seems much too strong, but intuitions aren't clear.

6.4 Prediction: SMs under negation are weird

(51) a. Policemen rarely carry more than one gun. (Bart Geurts)
b. ??Policemen rarely carry at least one gun.

(52) a. There was so much gold there, that no man found less than ten grams.
b. ??There was so much gold thhere, that no man found at most ten grams.

$\neg(\Diamond(A \vee B) \wedge \neg(\Diamond(C \vee D)))$ is e.g. verified by $\Diamond(C)$.

So "*This policemen doesn't carry at least one gun*" is verified by a policeman that possibly carries no gun.

7 Conclusions

▷ *At least two* and *more than one* are not interchangeable.

▷ CMs have a relatively simple semantics that interacts with a scale of alternatives made salient by focus.

▷ SMs are interpreted with respect to a modal.

▷ SM interpretation is strong.

▷ The interaction with modality predicts that SMs are unfit to express certainty and that they are difficult to embed under negation.

Some problems remain:

▷ I have remained vague about the underlying theory of scales of alternatives. (How) do scales enter in compositional interpretation?

▷ I have remained vague about the strong interpretation of numerals (the exactly n reading). What does the compositional process look like exactly?

▷ Intuitions are difficult for some cases.

References

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